

## WEAK APPROXIMATIONS FOR WIENER FUNCTIONALS

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ABSTRACT. In this paper we introduce a simple space-filtration discretization scheme on Wiener space which allows us to study weak decompositions of a large class of Wiener functionals. We show that any Wiener functional has an underlying robust semimartingale skeleton which under mild conditions converges to it. The approximation is given in terms of discrete-jumping filtrations which allow us to approximate irregular processes by means of a stochastic derivative operator on Wiener space introduced in this work. As a by-product, we prove that continuous paths and a suitable notion of energy are sufficient in order to get a unique orthogonal decomposition similar to weak Dirichlet processes. In this direction, we generalize the main results given in Graversen and Rao [24] and Coquet et al. [11] in the particular Brownian filtration case.

The second part of this paper is devoted to the application of these abstract results to concrete irregular processes. We show that our embedded semimartingale structure allows a very explicit and sharp approximation scheme for densities of square-integrable Brownian martingales in full generality. In the last part, we provide new approximations for integrals w.r.t the Brownian local-time.

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*Date:* September 9, 2009.

*1991 Mathematics Subject Classification.* Primary: 60Hxx; Secondary: 60H20.

*Key words and phrases.* Wiener functionals, weak Dirichlet processes, Clark-Ocone formula, stochastic derivatives.

The research of A. Ohashi was supported in part by FAPESP grant no. 05/57064-4.

## 1. INTRODUCTION AND DISCUSSION OF THE MAIN RESULTS

The main goal of this article is to provide a very explicit discretization scheme in order to approximate *irregular* cadlag processes<sup>1</sup> adapted to the Brownian filtration (henceforth abbreviated by Wiener functional). We are interested in providing an underlying robust semimartingale skeleton which under mild and readable assumptions converges in a weak sense to the original process. In this section, we give an overview of the main results without giving all the required technical assumptions.

Discretization methods for stochastic systems have been always a topic of great interest in stochastic analysis and its applications. Since the pioneering work of Wong and Zakai we know that not every choice of discretization procedure will lead to good stability properties of elementary processes such as Itô integrals and related stochastic equations. In this direction, motivated by stability properties of backward stochastic differential equations and discrete-time approximations for continuous-time financial models, a number of works in the last few years have been written on theorems like “if  $(H^n, W^n) \rightarrow (H, W)$  then  $\int H^n dW^n \rightarrow \int H dW$  or if  $W^n \rightarrow W$  then  $X^n = \int f(X^n) dW^n \rightarrow X = \int f(X) dW$ ”. See the works [38, 31, 27, 32, 12, 4] and other references therein.

In order to get those convergence results one has to assume suitable compactness arguments which allow one to exchange the limits. In one hand, one may interpret such assumptions as simple technical arguments imposed on the system to get the desirable robustness. On the other hand, Graversen and Rao [24] have shown a closed relation between finite energy and the existence of Dobb-Meyer-type decompositions. More recently, Coquet et al [11] has proved the uniqueness of such decompositions by means of the so-called weak Dirichlet processes which can be expressed (uniquely) as the sum of a local martingale and a process which is orthogonal to local martingales. Introduced by Errami and Russo [19] with later developments discussed by Coviello and Russo [13] and Gozzi and Russo [23], such class of processes constitutes a natural generalization of Dirichlet processes, which in turn naturally extends semimartingales.

While the abstract relation between energy (under deterministic partitions) and martingale decompositions of Wiener functionals seems to be well-understood, the results towards the obtention of fairly explicit and robust semimartingale approximating sequences by means of such types of compactness arguments is unsatisfactory. In this direction, one abstract question would be: Under which mild and readable conditions one may find a *robust* approximating sequence of *smooth* semimartingales which converges to an irregular Wiener functional? Moreover, can we obtain new classes of processes which can be fairly approximated by semimartingales? This is the program that we start to carry out in this article.

In this work, we are mainly interested in providing a *strongly robust* discretization scheme in a very constructive way which allows us to approximate a given Wiener functional. In order to illustrate the basic idea, let us assume that a given process  $X$  has an abstract representation

$$(1.1) \quad X_t = X_0 + \int_0^t H_s dB_s + N_t,$$

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<sup>1</sup>Processes with right-continuous paths and with finite left hand limits.

where  $B$  is the Brownian motion under its natural filtration  $\mathbb{F}$ ,  $N$  may be a discontinuous non-semimartingale process and  $H$  is a progressive process which is completely unknown a priori. One of the problems addressed in this paper is the following one: Construct an explicit sequence of  $\mathbb{F}^k$ -special semimartingales  $(X^k, M^k, N^k)$  given by

$$X^k = X_0 + \int H^k dA^k + N^k, \quad M^k = \int H^k dA^k, \quad \mathbb{F}^k \subset \mathbb{F},$$

where  $H^k$  is fully based on the information generated by  $X$  such that

$$H^k \rightarrow H, \quad A^k \rightarrow B, \quad \int H^k dA^k \rightarrow \int H dB, \quad N^k \rightarrow N, \quad \mathbb{F}^k \rightarrow \mathbb{F}.$$

The main difficulty in answering this question comes from the fact that when  $X$  is very rough the joint convergence of  $(H^k, \int H^k dA^k)$  to  $(H, \int H dB)$  in general will not hold since  $H$  has no a priori path regularity. Similarly,  $N$  may be very irregular in such way that  $N^k \rightarrow N$  will not hold either. A similar type of problem was addressed by Jacod et al. [27] in a pure martingale and Markovian setup at a fixed terminal time  $0 < T \leq \infty$ . In [27], they have provided reasonable explicit expressions for  $H^k$  when  $B$  and  $N$  are replaced by orthogonal square-integrable martingales w.r.t an arbitrary filtration. More explicit expressions were proved by imposing an underlying Markovian structure.

In this paper, we are interested in somehow more irregular objects such as non-Markovian and non-semimartingale processes restricted to the Wiener space. A typical example that we have in mind is given by  $\mathbb{E}[D_t Y | \mathcal{F}_t]$  where  $Y$  is a square integrable random variable and  $D$  denotes the Gross-Sobolev derivative. Other examples of irregular processes that we are interested in this paper are given by the local-time integrals  $\int_{\mathbb{R}} f(x) d_x L_t^x$  where  $f \in L_{loc}^2(\mathbb{R})$  and  $L_t^x$  is the Brownian local-time. Processes of type  $f(B^H)$  arising from the fractional Brownian motion  $B^H$  with  $1/2 < H < 1$  are also studied.

In order to study Wiener functionals of type (1.1), an abstract theory is developed based on an underlying smooth semimartingale skeleton. Such embedded structure is generated by a space-filtration discretization scheme induced by a suitable sequence of stopping times which measures the instants when the Brownian motion reaches some a priori levels. Two fundamental notions play a key rule in our results: Energy and covariations computed only in terms of the information generated by the Brownian motion on such stopping times. Generally speaking, we say that a given processes  $X$  has finite energy if

$$(1.2) \quad \sup_n \mathbb{E} \sum_{t_i^n \in D_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 < \infty,$$

for a given deterministic refining sequence  $D_n$  of subdivisions of  $[0, T]$  whose mesh goes to zero. The main result in [24] states that if  $X$  has finite energy then there exists at least one decomposition

$$(1.3) \quad X = X_0 + M + N,$$

where  $M$  is a local martingale and  $N$  is a predictable process which is orthogonal in some sense to martingales. Beyond the class of semimartingales, decompositions of

type (1.3) has been studied by several authors under different types of covariations whose the notion of orthogonality is computed. The first attempt to give a meaning to quadratic variations for a class of processes beyond semimartingales is due to Folmer [21, 22] by means of time-discretizations. A different type of covariation notion can also be considered via regularization which proves to be a fruitful way towards a non-semimartingale stochastic analysis. See Russo and Vallois [37] for a recent survey on this topic.

In order to approximate a given irregular process of type (1.1) in a very explicit and robust way, we replace the classical notions of energy (1.2) and covariation by means of a space-filtration discretization scheme which can be heuristically described as follows. For each  $k \geq 1$ , let  $(T_n^k)_{n \geq 0}$  be the sequence of stopping times defined in (2.1). Let  $\mathbb{F}^k$  be the filtration generated by the pure-jump process  $A_t^k = B_{T_n^k}$  for  $\{T_n^k \leq t < T_{n+1}^k\}$ . If  $X$  is a given Wiener functional, let  $\delta^k X$  be the natural embedded semimartingale skeleton based on  $\mathbb{F}^k$  (see (2.10)). The following two quantities will play a crucial role in this work. The classical notion of energy (1.2) is replaced by

$$(1.4) \quad \mathcal{E}_2(X) = \sup_{k \geq 1} \mathbb{E}[\delta^k X, \delta^k X]_T,$$

where  $[\cdot, \cdot]$  denotes the usual covariation between semimartingales. If  $X$  and  $Y$  are two Wiener functionals, we say that  $X$  and  $Y$  admits the  $\delta$ -covariation if

$$(1.5) \quad \langle X, Y \rangle^\delta = \lim_{k \rightarrow \infty} [\delta^k X, \delta^k Y].$$

exists in a weak sense. Contrary to previous works where the concept of covariation and energy are computed via time-discretization or regularization, the objects given in (1.4) and (1.5) can be viewed in a pure space-filtration discretization setup. In the sequel, we denote by  $H^2$  the space of square-integrable Brownian martingales starting from zero. The main results of this article are summarized by the following theorem:

**Theorem 1.1.** *Assume that a given Wiener functional  $X$  satisfies  $\mathcal{E}_2(X) < \infty$ ,  $\lim_{k \rightarrow \infty} \delta^k X = X$  weakly and  $\langle X, B \rangle^\delta$  exists. Then there exists a unique orthogonal decomposition*

$$(1.6) \quad X = X_0 + M^X + N^X,$$

where  $M^X \in H^2$  and  $\langle N^X, W \rangle^\delta = 0$  for every  $W \in H^2$ . Moreover, there exists a pre-limit semimartingale sequence  $(M^{k,X}, N^{k,X}) \rightarrow (M^X, N^X)$  given by (2.16) in a weak sense. In addition, if  $M^X = \int H^X dB$  then

$$(1.7) \quad \mathcal{D}X := \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[} = H^X \quad \text{weak sense,}$$

where  $h^k$  is a regularizing sequence of deterministic smooth functions given in (2.8).

The existence of the orthogonal martingale decomposition (1.6) in Theorem 1.1 is very reminiscent from the early work of Graversen and Rao [24] and Coquet

et al. [11]. The main novelty of Theorem 1.1 is the obtention of readable sufficient conditions which allow us to find an explicit pre-limit robust semimartingale sequence (2.16) converging to a given Wiener functional in terms of a stochastic derivative operator (1.7). Moreover, the assumptions in Theorem 1.1 are rather weak in the sense that a mild path regularity in addition to finite energy are sufficient in order to approximate a given Wiener functional as the next theorem shows. In the sequel,  $H_{loc}^2$  stands for the local version of  $H^2$ .

**Theorem 1.2.** *Assume that a given continuous Wiener functional  $X$  admits  $\mathcal{E}_2(X) < \infty$  locally. Then  $\langle X, B \rangle^\delta$  exists locally and therefore there exists a unique decomposition*

$$(1.8) \quad X = X_0 + M + V$$

where  $M \in H_{loc}^2$  and  $V$  is  $\delta$ -locally orthogonal to  $H^2$ . Moreover,  $X$  can be approximated (locally) by the sequence of semimartingales  $(M^{k,X}, N^{k,X})$  in (2.16).

The proofs of Theorems 1.1 and 1.2 will be given in Sections 3, 4 and 5. In Theorems 1.1 and 1.2, the  $\delta$ -orthogonality relation means  $\langle Y, W \rangle^\delta = 0$  for every  $W \in H^2$ . A few words about the above theorems are convenient at this stage. Contrary to previous works where the notion of finite energy (1.2) and continuous paths do not ensure a unique orthogonal martingale decomposition, Theorem 1.2 guarantees uniqueness by taking advantage of the underlying smooth semimartingale skeleton induced by our space-filtration discretization scheme. For instance, neither Hölder paths nor strong predictability (see [13]; Corollary 3.14) are assumed in Theorem 1.2 in order to get the orthogonal decomposition (1.8). Moreover, in general the  $\delta$ -covariation (1.5) does not coincide with the ones given by Folmer [22], Bertoin [6], Coquet et al [11] and Russo and his collaborators [19, 13, 37], but it coincides on the class of semimartingales. By combining Theorem 1.1 and Theorem 1.2, we not only give a general statement on the existence - uniqueness of martingale decompositions but we show that every Wiener functional has an embedded explicit semimartingale skeleton which under some mild conditions converges to it in a robust way.

In the particular pure martingale case, the following general approximation scheme holds for the classical Itô representation theorem. In the sequel,  $D$  stands for the Gross-Sobolev derivative on the Gaussian space of the Brownian motion.

**Corollary 1.1.** *If  $Y$  is an square-integrable random variable then*

$$(1.9) \quad \mathcal{D}Y = \mathbb{E}[DY|\mathcal{F}] = \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\mathbb{E}[Y|\mathcal{G}_n^k] - \mathbb{E}[Y|\mathcal{G}_{n-1}^k]}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[} \quad \text{weak sense,}$$

where  $\mathcal{G}_n^k$  are suitable finitely generated sigma-algebras. Therefore, the classical Clark-Ocone formula can be written as

$$Y = \mathbb{E}[Y] + \int_0^T \mathcal{D}_s Y dB_s.$$

Contrary to previous works (see, for instance, [16, 17, 35] and other references therein) the approximation scheme given in (1.9) is intrinsic and it is rather explicit

without imposing smoothness in the sense of Malliavin calculus and no underlying Markovian or continuous semimartingale structure is assumed (see e.g [1, 41]). We also have shown that for sufficiently smooth real functions  $f$ , the process  $f(B^H)$  satisfies the assumptions in Theorem 1.1 for  $1/2 < H < 1$  and therefore we provide an explicit smooth pre-limit semimartingale sequence converging to  $f(B^H)$ . See Proposition 6.1.

In the last part of this paper, we study the non-semimartingale term in the Itô formula given by  $\int_{-\infty}^{\infty} f(x) d_x L_t^x$  where  $f \in L_{loc}^2(\mathbb{R})$  and  $\{L_t^x; x \in \mathbb{R}\}$  is the Brownian local-time. It is well-known [36, 20, 18] that for any locally-square integrable function  $f$  it is possible to define

$$(1.10) \quad \int_{-\infty}^{\infty} f(x) d_x L_t^x = 2 \left[ \int_0^t f(B_s) dB_s - (F(B_t) - F(B_0)) \right],$$

where  $f$  is the primitive of  $F$ . Moreover,  $F(B)$  is a Dirichlet process with the above decomposition. In this paper, we give a new proof of this result by using the structural Theorems 1.1 and 1.2. By taking advantage of our space-filtration discretization scheme, one only has to show that  $F(B)$  has finite energy in the sense of (1.4), where upcrossing estimates play a key rule. The main result in this direction is the following approximation for integrals w.r.t the Brownian local time.

**Proposition 1.1.** *Assume that  $F$  is an absolutely continuous function with locally-square integrable derivative  $f$ . Then the expression (1.10) holds and if  $f \in L^2(\mathbb{R})$  we have*

$$-\int_{-\infty}^{\infty} f(x) d_x L_t^x = \lim_{k \rightarrow \infty} \frac{1}{2^{-2k}} \sum_{i=-1}^1 \int_0^t \{F(A_{s-}^k + i2^{-k}) - F(A_{s-}^k)\} h_s^k ds \quad \text{weakly.}$$

The proof of Proposition 1.1 is given in Section 7 and can be adapted to any square-integrable Brownian martingale by means of Theorem 1.2. This will be implemented elsewhere in a forthcoming paper.

This article is organized as follows. In Section 2, we fix the notation and we give some preliminary results regarding the pre-limit sequence and its basic properties. In Section 3, we establish the convergence of the semimartingale skeleton. In Section 4, we investigate basic properties of the resulting weak decompositions. In Section 5, we investigate the martingale representation in the weak decompositions by introducing a stochastic derivative w.r.t Brownian motion. In Section 6, some examples are investigated and we establish a change of variables formula for our basic processes. In Section 7, we investigate in detail the Itô formula for the Brownian motion in the light of the theory developed in this paper.

## 2. PRELIMINARIES

In this section, we fix the basic notation and framework that we use in this paper and present some elementary results concerning our approximation scheme. Throughout this paper we are given the usual stochastic basis  $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$  of the standard Brownian motion  $B$  starting from 0, where  $\Omega$  is the set  $\mathcal{C}(\mathbb{R}_+; \mathbb{R}) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ continuous}; f(0) = 0\}$ ,  $\mathcal{F}$  is the completed Borel sigma algebra,  $\mathbb{P}$  is the Wiener measure on  $\Omega$  and  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  is the usual  $\mathbb{P}$ -augmentation of the natural

filtration generated by the Brownian motion. We denote by  $\mathcal{O}$  the optional sigma algebra with respect to  $\mathbb{F}$ .

For each positive integer  $k$ , we define  $T_0^k = 0$  a.s and

$$(2.1) \quad T_n^k = \inf\{T_{n-1}^k < t < \infty; |B_t - B_{T_{n-1}^k}| = 2^{-k}\}, \quad n \geq 1.$$

One should notice that  $(T_n^k)_{n \geq 0}$  is a sequence of  $\mathbb{F}$ -stopping times for every  $k$ . Moreover,  $T_n^k < \infty$  a.s for every  $k$  and  $n$  such that  $T_n^k < \infty \Rightarrow T_n^k < T_{n+1}^k$  and  $T_n^k \uparrow \infty$  a.s. as  $n \rightarrow \infty$  for each  $k \geq 1$ . The  $(k+1)$ th stopping time sequence is a refinement of the  $k$ th in the sense that  $(T_n^k)_{n=0}^\infty$  is a subsequence of  $(T_n^{k+1})_{n=0}^\infty$  such that for any  $n \geq 0$  there exist  $j_1$  and  $j_2$ ,  $T_{j_1}^{k+1} = T_n^k$  and  $T_{j_2}^{k+1} = T_{n+1}^k$ . Moreover, if  $Z_k(n) = B_{T_n^k} - B_{T_{n-1}^k}$  then  $Z_k(1), Z_k(2), \dots$  are i.i.d. with  $\mathbb{P}(Z_k(1) = -2^{-k}) = \mathbb{P}(Z_k(1) = 2^{-k}) = 1/2$ , where the independence property yields the differences  $T_n^k - T_{n-1}^k$  are i.i.d as well with the same distribution as  $T_1^k$ .

Next we consider the following family of random variables

$$(2.2) \quad \sigma_n^k = \begin{cases} 1 & ; \quad B_{T_n^k} - B_{T_{n-1}^k} = 2^{-k} \text{ and } T_n^k < \infty \\ -1 & ; \quad B_{T_n^k} - B_{T_{n-1}^k} = -2^{-k} \text{ and } T_n^k < \infty \\ 0 & ; \quad T_n^k = \infty. \end{cases}$$

We then define the following family of step processes

$$A_t^k = \sum_{n=1}^{\infty} 2^{-k} \sigma_n^k \mathbb{1}_{\{T_n^k \leq t\}}.$$

Let  $(\mathcal{F}_t^k)_{t \geq 0}$  be the natural filtration generated by  $\{A_t^k; 0 \leq t < \infty\}$ . One should notice that  $(\mathcal{F}_t^k)_{t \geq 0}$  is a discrete-type filtration in the sense that

$$(2.3) \quad \mathcal{F}_t^k = \bigcup_{i=0}^{\infty} \left( \mathcal{G}_i^k \cap \{T_i^k \leq t < T_{i+1}^k\} \right), \quad t \geq 0,$$

where  $\mathcal{G}_0^k := \{\Omega, \emptyset\}$  and  $\mathcal{F}_{T_n^k}^k = \mathcal{G}_n^k := \sigma(T_1^k, \dots, T_n^k, \sigma_1^k, \dots, \sigma_n^k)$ . Moreover, since  $\mathcal{G}_n^k = \sigma(A_{s \wedge T_n^k}^k; s \geq 0)$  then  $\mathcal{G}_n^k$  and  $\mathcal{F}_t^k$  coincide up to  $\mathbb{P}$ -null sets on  $\{T_n^k \leq t < T_{n+1}^k\}$ . In other words,  $(\mathcal{F}_t^k)_{t \geq 0}$  is a jumping filtration (e.g [26]) with jumping sequence given by  $(T_n^k)_{n \geq 1}$  for each  $k \geq 1$ . With a slight abuse of notation we write  $\mathcal{F}_t^k$  to denote its  $\mathbb{P}$ -augmentation satisfying the usual conditions, where  $\mathbb{F}^k := (\mathcal{F}_t^k)_{t \geq 0}$ . We also denote by  $\mathcal{O}^k$  and  $\mathcal{P}^k$  the optional and predictable sigma algebras, respectively, with respect to  $\mathbb{F}^k$ .

In this work, the  $\mathbb{F}^k$ -predictable and optional projections of a given measurable process  $X$  will be denoted by  ${}^{p,k}[X]$  and  ${}^{o,k}[X]$ , respectively. The  $\mathbb{F}^k$ -dual predictable and optional projections will be denoted by  $[Y]^{p,k}$  and  $[Y]^{o,k}$ , respectively. We also denote by  $[X, Y]$  and  $\langle X, Y \rangle$  the usual quadratic variation and angle bracket of a pair of semimartingales, respectively. The usual jump of a process is denoted by  $\Delta Y_t = Y_t - Y_{t-}$  where  $Y_{t-}$  is the left-hand limit of a cadlag process  $Y$ . Moreover, if  $T$  and  $S$  are stopping times then  $[[T, S]]$ ,  $[[T, S[[$  and  $]]T, S]]$  will denote the usual stochastic intervals.

We now give some elementary properties of our discretization scheme.

**Lemma 2.1.**  $\{A_t^k; 0 \leq t < \infty\}$  has independent increments and it is a square-integrable  $\mathbb{F}^k$ -martingale with locally integrable variation. Moreover, for every  $0 < T < \infty$  we have

$$(2.4) \quad \sup_{0 \leq t \leq T} \|B_t - A_t^k\|_\infty \leq 2^{-k}, \quad k \geq 1,$$

where  $\|\cdot\|_\infty$  denotes the usual norm on the space  $L^\infty(\mathbb{P})$ .

*Proof.* The estimate (2.4), the independence of the increments and the locally integrable variation property are immediate consequences of the definitions. For the martingale property we notice from (2.3) that we can write

$$\mathcal{F}_t^k = \left\{ \bigcup_{i=0}^{\infty} A_n \cap [T_i^k \leq t < T_{i+1}^k]; A_n \in \mathcal{G}_n^k, n \geq 0 \right\}, \quad t \geq 0,$$

where  $A_s^k = B_{T_n^k}$  on  $[T_n^k \leq s < T_{n+1}^k]$  for each  $n \geq 1$ . In this case, the usual optional stopping theorem gives the following representation (see also Remark 2.3)

$$\mathbb{E}[B_t | \mathcal{F}_s^k] = A_s^k \quad a.s. \quad s \leq t,$$

and therefore we may conclude that  $A^k$  is an  $\mathbb{F}^k$ -martingale. The integrability property follows from the well-known fact that

$$\mathbb{E} \sum_{n=1}^{\infty} (B_{T_n^k} - B_{T_{n-1}^k})^2 \mathbb{1}_{\{T_n^k \leq T\}} < \infty.$$

□

The following result turns out to be very useful for the approach taken in this work. In the sequel, we denote by  $\pi$  the usual projection of  $\mathbb{R}_+ \times \Omega$  onto  $\Omega$ . For any measurable sets  $D$  and  $A$  we write  $D - A$  to denote  $D \cap A^c$ , where  $A^c$  is the complement of the set  $A$ . Moreover,  $\bigvee_{k \geq 0} \mathcal{A}_k$  denotes the sigma-algebra generated by  $\bigcup_{k \geq 0} \mathcal{A}_k$  for a sequence of classes  $\{\mathcal{A}_k; k \geq 0\}$ .

**Lemma 2.2.** *The natural filtration of  $A^k$  satisfies the following properties:*

(i)  $\mathbb{F}^k$  is an increasing family of sigma-algebras such that  $\mathcal{F}_t = \bigvee_{k \geq 0} \mathcal{F}_t^k$  for every  $t \geq 0$ .

(ii) The sequence of filtrations  $\mathbb{F}^k$  converges weakly to  $\mathbb{F}$ .

(iii) For every  $O \in \mathcal{O}$  there exists a sequence  $O^k \in \mathcal{O}^k$  such that

$$O^k \subset O \quad \forall k \geq 1, \quad \text{and} \quad \mathbb{P}[\pi(O) - \pi(O^k)] \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* It is straightforward to check that  $\mathcal{F}_t^k \subset \mathcal{F}_t^{k+1}$  for every  $k$  and  $t \geq 0$ . Moreover, each cylinder set of the form  $\{b_1 < B_t \leq b_2\}$  can be approximated by

$$(2.5) \quad \{b_1 + 2^{-k} < A_t^k \leq b_2 - 2^{-k}\} \subset \{b_1 < B_t \leq b_2\} \subset \{b_1 - 2^{-k} < A_t^k \leq b_2 + 2^{-k}\} \quad a.s.$$

Thus proving part (i). To prove part (ii) we only need to show that for each  $B \in \mathcal{F}_T$  the sequence of martingales  $\mathbb{E}[\mathbb{1}_B | \mathcal{F}_t^k]$  converges in probability to  $\mathbb{E}[\mathbb{1}_B | \mathcal{F}]$  on the



space of cadlag functions equipped with the usual Skorohod topology. But this is a simple application of ([10], Prop. 4). Now let us fix an arbitrary  $0 < t < \infty$ . From (2.5) we know that for any cylinder set restricted on  $[0, t]$  we may find two sequences  $(D_i^k)_{k \geq 1}$   $i = 1, 2$  such that

$$(2.6) \quad D_1^k \subset D \subset D_2^k, \quad \forall k \geq 1,$$

where  $D_1^k \subset D_1^{k+1}$  and  $D_2^k \supset D_2^{k+1}$  for every  $k \geq 1$ . From (2.4) it follows that

$$(2.7) \quad \max\{\mathbb{P}[D - D_1^k]; \mathbb{P}[D_2^k - D]\} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

In fact, by a standard monotone class argument one can easily show that any set in  $\mathcal{F}_t$  satisfies the above property. Now recall that  $\mathcal{O} = \sigma(\mathcal{C})$  where

$$\mathcal{C} = \{E \times \{0\} : E \in \mathcal{F}_0\} \cup \{[s, t) \times E : s < t, s, t \in \mathbb{Q}_+, E \in \mathcal{F}_s\}.$$

From (2.6) and (2.7) it follows that for each  $\Lambda \in \mathcal{C}$  there exists sequences  $O_i^k$   $i = 1, 2$  such that  $O_1^k \subset \Lambda \subset O_2^k$  and

$$\max\{\mathbb{P}[\pi(\Lambda) - \pi(O^k)]; \mathbb{P}[\pi(O^k) - \pi(\Lambda)]\} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Let  $O$  be an arbitrary optional set in  $\mathcal{O}$ . It is well known (e.g [25]; Th. 4.5) that for any  $\varepsilon/2 > 0$  there exists a sequence  $(B_n)_{n \geq 1}$  in  $\mathcal{C}$  such that  $D := \cap_{n \geq 1} B_n \subset O$  and  $\mathbb{P}[\pi(O) - \pi(D)] \leq \varepsilon/2$ . We may assume that  $B_i \supset B_{i+1}$  and they are given by

$$B_i = X_i \times [s_i, t_i); \quad X_i \in \mathcal{F}_{s_i}.$$

From the previous arguments there exists a sequence  $(E_i^k)_{i \geq 1}$  in  $\mathcal{P}^k$  such that  $E_i^k \subset B_i$ ,  $i, k \geq 1$ . Moreover, we may assume that  $E_i^k$  is given by the following generic form

$$E_i^k = \Lambda_i^k \times [s_i, t_i - a_i], \quad i \geq 1,$$

where in this case  $\lim_{k \rightarrow \infty} \mathbb{P}[X_i - \Lambda_i^k] = 0$  for every  $i \geq 1$ . Now one should notice that since  $[s_i, t_i - a_i]$  are compact sets then  $\cap_{i \geq 1} [s_i, t_i - a_i] \neq \emptyset$  and hence  $\pi(\cap_{n \geq 1} E_n^k) = \cap_{n \geq 1} \pi(E_n^k)$  for every  $k \geq 1$ . Therefore, it follows that

$$\mathbb{P}[\pi(\cap_{i=1}^n B_i) - \pi(\cap_{i=1}^n E_i^k)] = \mathbb{P}[\cup_{i=1}^n \pi(B_n) \cap \pi^c(E_i^k)] \leq \sum_{i=1}^n \mathbb{P}[\pi(B_n) - \pi(E_i^k)] \rightarrow 0,$$

as  $k \rightarrow \infty$ . Since the double limit  $\lim_{k, n \rightarrow \infty} \pi(\cap_{i=1}^n B_i) - \pi(\cap_{i=1}^n E_i^k)$  exists we may use the continuity of the measure to conclude that

$$\mathbb{P}[\pi(D) - \pi(\cap_{i \geq 1} E_i^k)] < \varepsilon/2,$$

for every  $k$  sufficiently large. By writing

$$\mathbb{P}[\pi(O)] - \mathbb{P}[\pi(\cap_{i \geq 1} E_i^k)] = \mathbb{P}[\pi(O)] - \mathbb{P}[\pi(D)] + \mathbb{P}[\pi(D)] - \mathbb{P}[\pi(\cap_{i \geq 1} E_i^k)],$$

we conclude the proof of the Lemma.  $\square$

In this work, the angle bracket  $\langle A^k, A^k \rangle$  process will play a key rule in our approximation scheme.

**Lemma 2.3.** *The angle bracket of  $A^k$  is a deterministic continuous process given by*

$$(2.8) \quad \langle A^k, A^k \rangle_t = \int_0^t h^k(s) ds,$$

where  $h^k$  is the intensity of the jump process  $[A^k, A^k]_t = 2^{-2k} \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n^k \leq t\}}$ . Moreover,

$$(2.9) \quad \sup_{0 \leq t \leq T} |\langle A^k, A^k \rangle_t - t| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* It is well known that  $\{(T_n^k - T_{n-1}^k); n \geq 1\}$  is a sequence of i.i.d random variables with the same distribution of  $2^{-2k}\tau$ , where  $\tau = \inf\{t > 0 : |B_t| = 1\}$  is an absolutely continuous random variable. Then, the point process  $[A^k, A^k]_t = 2^{-2k} \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n^k \leq t\}}$  has independent increments and the angle bracket  $\langle A^k, A^k \rangle_t$  is a deterministic absolutely continuous process. From (ii) in Lemma 2.2 and ([10], Th.3), we know that  $[A^k, A^k] \rightarrow [B, B]$  uniformly in probability as  $k \rightarrow \infty$ . Since  $[A^k, A^k]_T$  is uniformly integrable, the convergence stated in (2.9) holds.  $\square$

One should notice that since  $T_1^k$  is not exponentially distributed, the point process  $\sum_{n=1}^{\infty} \mathbb{1}_{\{T_n^k \leq t\}}$  cannot have stationary increments although it is an inhomogeneous Poisson process. Moreover, the deterministic kernel  $h^k$  in Lemma 2.3 can be easily computed in terms of the Laplace transform of  $\tau$ . See e.g [9] for more details.

**Remark 2.1.** *Since  $T_1^k$  is an absolutely continuous random variable and  $A^k$  is a point process it follows that  $\mathbb{F}^k$  is a quasi-left-continuous filtration. From [26] we know that every  $\mathbb{F}^k$  - martingale has jumping times given by the totally inaccessible stopping times  $(T_n^k)_{n \geq 1}$ . Moreover,  $\mathbb{F}^k$  supports only martingales of bounded variation where  $\mathbb{F}^k$  - local martingales are purely discontinuous.*

In the remainder of this paper, we will adopt the following terminology.

**Definition 2.1.** *We say that  $X$  is a **Wiener functional** if it is  $\mathbb{F}$ -adapted and it has cadlag paths.*

**Remark 2.2.** *Throughout this paper, if  $X$  is a Wiener functional then we assume that  $\mathbb{E}|X_{T_n^k}| < \infty$  for every  $k, n \geq 1$ .*

We now embed a given Wiener functional  $X$  into a sequence of  $\mathbb{F}^k$  - quasi-left continuous bounded variation processes as follows

$$(2.10) \quad \delta^k X_t := X_0 + \sum_{n=1}^{\infty} \mathbb{E}[X_{T_n^k} | \mathcal{G}_n^k] \mathbb{1}_{\{T_n^k \leq t < T_{n+1}^k\}}.$$

**Remark 2.3.** (a) *The usual optional stopping theorem implies that any  $\mathbb{F}$  - martingale  $M$  with  $M_0 = 0$  a.s admits the following representation*

$$(2.11) \quad \delta^k M_t = \mathbb{E}[M_T | \mathcal{F}_t^k] \quad 0 \leq t \leq T.$$

*In particular,  $A^k = \delta^k B$ .*

Next, our goal is to establish an explicit decomposition for the embedded semimartingale skeleton  $(\delta^k X)_{k \geq 1}$  in terms of a discrete-type derivative.

**2.1. The approximate decomposition.** In this section, we obtain an explicit Doob-Meyer decomposition for  $\delta^k X$ . The following remark is a straightforward consequence of the definitions and the quasi-left continuity of  $\delta^k X$ .

**Lemma 2.4.** *The process  $\{\delta^k X_t : 0 \leq t \leq T\}$  is an  $\mathbb{F}^k$ -adapted process with locally integrable variation for each  $k \geq 1$ . Moreover, there exists a unique  $\mathbb{F}^k$ -predictable process  $N^{k,X}$  with locally integrable variation such that*

$$(2.12) \quad \delta^k X_t - X_0 - N_t^{k,X} =: M_t^{k,X}$$

*is an  $\mathbb{F}^k$ -local martingale. The process  $N^{k,X}$  is the  $\mathbb{F}^k$ -dual predictable projection of  $\delta^k X_t - X_0$  which can be taken with continuous paths.*

From equation (2.12) it follows that  $\delta^k X$  is a special  $\mathbb{F}^k$ -semimartingale and therefore the above decomposition is the canonical one. Next we aim at characterizing the elements of the decomposition (2.12). One should notice that since  $\mathbb{F}^k$  is not a completely continuous filtration (see (2.2)), then  $A^k$  cannot have a strong predictable representation. However, by the independence of its increments it is well-known that  $A^k$  has the so-called optional representation (see e.g [25], Ex 13.9). That is, every  $\mathbb{F}^k$ -local martingale starting from zero is represented by an optional integral w.r.t  $A^k$ .

In the remainder of this paper, we make use of the optional stochastic integration w.r.t  $A^k$ . We refer the reader to [14, 25] for all details on optional integrals used in this paper. We just want to mention here that since the filtration  $\mathbb{F}^k$  is quasi-left continuous then the related optional integrals admit the usual operational properties of stochastic integrals with predictable integrands (see e.g. remark 35 in [14], page 346). In this work, we denote by  $\oint_0^t Y_s dA_s^k$  the optional integral of an  $\mathbb{F}^k$ -optional process  $Y$ .

We now introduce a process which will play a key rule in this work. If  $\delta^k X$  is the  $\mathbb{F}^k$ -projection of a Wiener functional  $X$ , then we define the following  $\mathbb{F}^k$ -optional process

$$(2.13) \quad \mathcal{D}\delta^k X := \sum_{n=1}^{\infty} \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} \mathbb{1}_{[[T_n^k, T_n^k]]}.$$

If

$$(2.14) \quad \mathbb{E} \sum_{n=1}^m |\Delta \delta^k X_{T_n^k}|^2 < \infty \quad \forall m, k \geq 1,$$

then

$$\left[ \int_0^\cdot \mathcal{D}_s^2 \delta^k X d[A^k, A^k]_s \right]^{1/2} = \left[ \sum_{n=1}^{\infty} \left( \delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k} \right)^2 \mathbb{1}_{\{T_n^k \leq \cdot\}} \right]^{1/2}$$

is a locally integrable increasing process for every  $k \geq 1$ . In this case, there exists a unique  $\mathbb{F}^k$ -local martingale  $M$  such that for every bounded  $\mathbb{F}^k$ -martingale  $V$ , the process  $[M, V] - \int_0^\cdot \mathcal{D}\delta^k X d[V, A^k]$  is an  $\mathbb{F}^k$ -local martingale and

$$\begin{aligned}
M_t &= \int_0^t \mathcal{D}_s \delta^k X dA_s^k - \left[ \int_0^\cdot \mathcal{D}_s \delta^k X dA_s^k \right]_t^{p,k} \\
&= \oint_0^t \mathcal{D}_s \delta^k X dA_s^k
\end{aligned}$$

where  $\int_0^t \mathcal{D}_s \delta^k X dA_s^k$  is interpreted in the Lebesgue-Stieltjes sense.

Now let us write the decomposition (2.12) in a more convenient way. Since  $M^{k,X}$  is a local-martingale with locally integrable variation we can represent it as a compensated sum of jumps as follows

$$M_t^{k,X} = Y_t^k - [Y^k]_t^{p,k}; \quad M_0^{k,X} = 0,$$

where  $Y_t^k = \sum_{0 \leq s \leq t} \Delta \delta^k X_s = \sum_{0 \leq s \leq t} \Delta M_s^{k,X}$  and  $[Y^k]^{p,k}$  can be taken with continuous paths. Of course, by the quasi left-continuity of  $\delta^k X$  one should notice that  $N_t^{k,X} = [Y^k]_t^{p,k}$ .

Now let us consider the following representation

$$Y_t^k = \sum_{n=1}^{\infty} \frac{\Delta \delta^k X_{T_n^k}^k}{\Delta A_{T_n^k}^k} \Delta A_{T_n^k}^k \mathbb{1}_{\{T_n^k \leq t\}}, \quad 0 \leq t \leq T.$$

In other words, we may write  $Y_t^k$  in terms of the following Lebesgue-Stieltjes integral

$$Y_t^k = \int_0^t \mathcal{D}_s \delta^k X dA_s^k.$$

Summing up the above arguments, we conclude that if  $X$  satisfies (2.14) then we arrive at the following optional representation for the martingale part in the decomposition (2.12)

$$M_t^{k,X} = \oint_0^t \mathcal{D}_s \delta^k X dA_s^k; \quad 0 \leq t \leq T.$$

Of course,  $\mathcal{D} \delta^k X$  is the unique  $\mathbb{F}^k$ -optional process which represents the martingale  $M^{k,X}$  as an optional stochastic integral with respect to the martingale  $A^k$ . Let us characterize the remainder term in the decomposition (2.12).

**Lemma 2.5.** *The  $\mathbb{F}^k$ -dual predictable projection of  $\delta^k X - X_0$  is given by the continuous process*

$$\int_0^t U_s^{k,X} d\langle A^k, A^k \rangle_s, \quad 0 \leq t \leq T,$$

where  $U^{k,X} := \mathbb{E}_{[A^k]}[\mathcal{D} \delta^k X / \Delta A^k | \mathcal{P}^k]$ . Here  $\mathbb{E}_{[A^k]}[\cdot | \mathcal{P}^k]$  denotes the conditional expectation with respect to  $\mathcal{P}^k$  under the Doléans measure generated by  $[A^k, A^k]$ . Moreover,

$$(2.15) \quad U_t^{k,X} = 0 \mathbb{1}_{\{T_0^k=t\}} + \frac{1}{2^{-2k}} \sum_{n=1}^{\infty} \mathbb{E}[X_t - X_{T_{n-1}^k} | \mathcal{G}_{n-1}^k; T_n^k = t] \mathbb{1}_{\{T_{n-1}^k < t \leq T_n^k\}}$$

*Proof.* Let us write  $Y_t^k = \int_0^t \mathcal{D}_s \delta^k X dA_s^k$  in the Lebesgue-Stieltjes sense. By the very definition

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{C\}}(s) dY_s^k \right] &= \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{C\}}(s) \mathcal{D}_s \delta^k X dA_s^k \right] = \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{C\}}(T_n^k) \mathcal{D}_{T_n^k} \delta^k X \Delta A_{T_n^k}^k \right] = \\ &= \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{C\}}(T_n^k) \frac{\mathcal{D}_{T_n^k} \delta^k X}{\Delta A_{T_n^k}^k} \Delta A_{T_n^k}^k \Delta A_{T_n^k}^k \right] = \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{C\}}(s) \frac{\mathcal{D}_s^k \delta^k X}{\Delta A_s^k} d[A^k, A^k]_s \right] = \\ &= \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{C\}}(s) U^{k,X}(s) d[A^k, A^k]_s \right] = \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{C\}}(s) U^{k,X}(s) d\langle A^k, A^k \rangle_s \right] \end{aligned}$$

for every  $C \in \mathcal{P}^k$ . This concludes the first part of the proof. Let us now consider the following sequence of sigma-algebras  $\mathcal{G}_{n-}^k := \mathcal{G}_{n-1}^k \vee \sigma(T_n^k)$ ,  $n \geq 1$ . We recall that for every  $C \in \mathcal{G}_{n-}^k$ , there exists a predictable process  $H$  such that  $H(T_n^k) = \mathbb{1}_C$  and it is null outside the stochastic interval  $]T_{n-1}^k, T_n^k]$  (see [8], Th. 31, pp. 307). Then, it follows from the first part that

$$\begin{aligned} \mathbb{E} [\mathbb{1}_C \Delta \delta^k X_{T_n^k} \mathbb{1}_{\{T_n^k \leq T\}}] &= \mathbb{E} \int_0^T H(s) dY_s^k \\ &= \mathbb{E} \int_0^T H(s) U^{k,X}(s) d[A^k, A^k]_s \\ &= \mathbb{E} [\mathbb{1}_C U^{k,X}(T_n^k) 2^{-2k} \mathbb{1}_{\{T_n^k \leq T\}}]. \end{aligned}$$

Since  $C$  is arbitrary and  $U^{k,X}$  a predictable process, it follows that

$$\begin{aligned} \mathbb{E} [\Delta \delta^k X_{T_n^k} \mathbb{1}_{\{T_n^k \leq T\}} \mid \mathcal{G}_{n-}^k] &= \mathbb{E} [U^{k,X}(T_n^k) 2^{-2k} \mathbb{1}_{\{T_n^k \leq T\}} \mid \mathcal{G}_{n-}^k] \\ &= U^{k,X}(T_n^k) 2^{-2k} \mathbb{1}_{\{T_n^k \leq T\}}. \end{aligned}$$

Then, one version of the conditional expectation can be written as follows

$$U^{k,X}(t) \mathbb{1}_{\{T_{n-1}^k < t \leq T_n^k \leq T\}} = \mathbb{E} \left[ \frac{\Delta \delta^k X_{T_n^k}}{2^{-2k}} \mid \mathcal{G}_{n-1}^k; T_n^k = t \right] \mathbb{1}_{\{T_{n-1}^k < t \leq T_n^k \leq T\}}.$$

This concludes the proof of the Lemma.  $\square$

Summing up all previous results of this section, we then arrive at the following representation.

**Proposition 2.1.** *If  $X$  is a Wiener functional satisfying assumption (2.14), then the unique  $\mathbb{F}^k$ -special semimartingale decomposition  $(M^{k,X}, N^{k,X})$  in (2.12) is actually given by*

$$(2.16) \quad \delta^k X_t = X_0 + \oint_0^t \mathcal{D}_s \delta^k X dA_s^k + \int_0^t U_s^{k,X} h_s^k ds,$$

where the kernel  $U^{k,X}$  is given by the expression (2.15).

### 3. WEAK DECOMPOSITION OF WIENER FUNCTIONALS

In this section, we are interested in providing readable conditions on a given Wiener functional  $X$  in such way that

$$X = \lim_{k \rightarrow \infty} \delta^k X; \quad M^X = \lim_{k \rightarrow \infty} M^{k,X}; \quad N^X = \lim_{k \rightarrow \infty} N^{k,X},$$

in a suitable topology.

Under such assumptions, we are able to decompose  $X$  into a unique orthogonal decomposition

$$X_t = X_0 + M_t^X + N_t^X,$$

where  $M^X$  is a martingale and  $N_t^X$  is an adapted process whose a specific type of *covariation* (See Definition 3.2) w.r.t Brownian martingales is null. Our main concern here is the study of such approximation in a very constructive and robust way, where the progressive process  $H^X$  which represents  $M^X$  must be given by a well-defined *derivative operator* w.r.t Brownian motion (see Section 5). Moreover, in Section 4 we prove that the  $\mathbb{F}^k$ -embedded semimartingale structure introduced in the last section allows us to obtain orthogonal decompositions for any continuous Wiener functional under a compactness assumption.

**3.1. Weak convergence and primary decomposition.** In this section, we investigate the convergence of our preliminary decomposition (2.16) given in terms of the approximation scheme  $(A^k, \mathbb{F}^k)$ . By carefully choosing a suitable topology on the space of processes, our strategy will be fully based on the information given by the quadratic variation of the martingale part in the same spirit of [24].

One should notice that we need to choose an appropriate topology which has to be general enough to afford the non-martingale part in the decomposition (2.16), and at the same time takes into account the desired convergence of the  $\mathbb{F}^k$  - local martingales  $M^{k,X}$  to an  $\mathbb{F}$  - local martingale.

Let  $B^p(\mathbb{F})$  be the set of all  $\mathbb{F}$  - adapted process with cadlag paths and which are  $1 \leq p < \infty$  Bochner integrable in the sense that

$$(3.1) \quad \|X\|_{B^p}^p = \mathbb{E}|X_T^*|^p < \infty,$$

where  $X_T^* := \sup_{0 \leq t \leq T} |X_t|$ . Of course,  $B^p(\mathbb{F})$  endowed with the norm  $\|\cdot\|_{B^p}$  is a Banach space, where the subspace  $H^p(\mathbb{F})$  of  $\mathbb{F}$  - martingales starting from zero is closed. For us the most important case will be  $p = 1$ . Even in this case, the above topology is quite strong to attend our needs. Recall that the topological dual  $M^\infty$  of  $B^1(\mathbb{F})$  is the space of processes  $A = (A^{pr}, A^{pd})$  such that

(i)  $A^{pr}$  and  $A^{pd}$  are right-continuous of bounded variation such that  $A^{pr}$  is  $\mathbb{F}$  - predictable with  $A_0^{pr} = 0$  and  $A^{pd}$  is  $\mathbb{F}$  - optional, purely discontinuous, respectively.

(ii)  $Var(A^{pd}) + Var(A^{pr}) \in L^\infty$ ,

where  $Var(\cdot)$  denotes the total variation of a bounded variation process on the interval  $[0, T]$ . The space  $M^\infty$  has the strong topology given by

$$\|A\|_{M^\infty} := \max\{\|Var(A^{pr})\|_\infty, \|Var(A^{pd})\|_\infty\}.$$

The duality pair is given by

$$(A, X) := \mathbb{E} \int_0^T X_s dA_s^{pr} + \mathbb{E} \int_0^T X_s dA_s^{pd}; \quad X \in B^1(\mathbb{F}),$$

where the following obvious estimate holds

$$|(A, X)| \leq \|A\|_{M^\infty} \|X\|,$$

for every  $A \in M^\infty$ ,  $X \in B^1(\mathbb{F})$ . We denote  $\sigma(B^1, M^\infty)$  the weak topology of  $B^1(\mathbb{F})$ . See the works ([14],[15],[33]) for a detailed discussion on the weak topology of  $B^1(\mathbb{F})$  restricted to the subspace of martingales  $H^1(\mathbb{F})$ .

In this article, it will be also useful to work with the following notion of convergence. Actually, one can show that the set  $\Lambda^\infty$  of the  $\mathbb{F}$  - optional bounded variation processes of the form

$$C = g \mathbb{1}_{\{S \leq \cdot\}}; \quad g \in L^\infty(\mathcal{F}_S), \quad S \text{ is an } \mathbb{F} - \text{stopping time (bounded by } T),$$

fulfills the Banach space  $B^1(\mathbb{F})$  in the sense that

$$(3.2) \quad \|X\|_{B^1} = \sup\{|(X, C)|; C \in \Lambda^\infty, \|C\|_{M^\infty} \leq 1\}.$$

Relation (3.2) is given in ([15]; Lemma 1) and therefore we may also endow  $B^1(\mathbb{F})$  with the  $\sigma(B^1, \Lambda^\infty)$  - topology induced by the family of seminorms

$$X \mapsto (X, C); \quad C \in \Lambda^\infty.$$

**Remark 3.1.** Actually, in [15] they have proved relation (3.2) on a larger space of the (only) measurable processes  $X$  such that  $\mathbb{E} \sup_{0 \leq t \leq T} |X_t| < \infty$ . Since  $B^1(\mathbb{F})$  is a closed subspace on it, then one can easily check (3.2) still holds. Obviously,  $\sigma(B^1, \Lambda^\infty)$  is weaker than  $\sigma(B^1, M^\infty)$ . However, relation (3.2) says that  $\Lambda^\infty$  is a norming subset of  $M^\infty$  and therefore  $\Lambda^\infty$  is  $w^*$  - dense in  $M^\infty$ .

**Remark 3.2.** A result due to Mokobodzki [33] states that if  $X^n$  is a sequence of optional processes such that  $\sup_{0 \leq t \leq T} |X_t^n|$  is uniformly integrable and for every  $S$  stopping time the sequence  $X_S^n$  converges weakly in  $L^1$  relatively to  $\mathcal{F}_S$ , then there exists an optional process  $X$  such that  $X^n \rightarrow X$  in  $\sigma(B^1, M^\infty)$ . As a consequence, if  $X^n \rightarrow X$  in  $\sigma(B^1, \Lambda^\infty)$  and  $\sup_{0 \leq t \leq T} |X_t^n|$  is uniformly integrable we do have convergence in  $\sigma(B^1, M^\infty)$ . See also Dellacherie et al. [15] for more details.

In the remainder of this paper, we shall write  $B^p$  ( $H^p$ ) to denote the space of Bochner integrable process ( $p$ -integrable martingales starting from zero) satisfying (3.1) endowed with the Brownian filtration  $\mathbb{F}$ . We now introduce the following quantity which will play a crucial role in this work.

**Definition 3.1.** We say that a given Wiener functional  $X$  has  **$p$ -finite energy** along the filtration family  $(\mathbb{F}^k)_{k \geq 1}$  if

$$(3.3) \quad \mathcal{E}_p(X) := \sup_{k \geq 1} \mathbb{E} \sum_{n=1}^{\infty} |\Delta \delta^k X_{T_n^k}|^p \mathbb{1}_{\{T_n^k \leq T\}} < \infty, \quad 1 < p < \infty.$$

**Remark 3.3.** *The above definition is similar in spirit to the classical notion of energy (e.g [24],[11]) for  $p = 2$ , but with one fundamental difference: The relevant information contained in the energy of  $X$  comes only from the sigma-algebras  $\mathcal{G}_n^k$  which reveal the information generated by the jumps of the projected Brownian motion  $A^k$  up to the stopping time  $T_n^k$ . Moreover, the path continuity of the predictable part  $N^{k,X}$  in the decomposition of the special semimartingale  $\delta^k X$  yields*

$$\mathcal{E}_2(X) = \sup_{k \geq 1} \mathbb{E}[M^{k,X}, M^{k,X}]_T.$$

In the remainder of this paper, if  $\mathcal{E}_2(X) < \infty$  then we say that  $X$  has finite energy.

We are now in position to study convergence of the decomposition given in (2.16). In the sequel, we fix an element  $X \in B^1$  and let  $(M^{k,X}, N^{k,X})$  be the associated canonical decomposition expressed in (2.16). Let us introduce the following family of  $\mathbb{F}$  - martingales

$$(3.4) \quad Z_t^{k,X} := \mathbb{E}[M_T^{k,X} | \mathcal{F}_t]; \quad 0 \leq t \leq T; k \geq 1.$$

In order to prove convergence of  $M^{k,X}$  to an  $\mathbb{F}$  - martingale we may use some standard compactness arguments. For the sake of completeness we give the details here.

**Lemma 3.1.** *The family of random variables  $\{[M^{k,X}, M^{k,X}]_T^{1/2} : k \geq 1\}$  is uniformly integrable if, and only if, the sequence of stochastic process  $\{Z^{k,X} : k \geq 1\}$  is weakly relatively compact in  $H^1$ .*

*Proof.* For any Young moderate function  $\phi$ , the Doob and Burkholder inequalities yield

$$\begin{aligned} \mathbb{E} \left[ \phi \left( \sup_{t \geq 0} |Z_t^{k,X}| \right) \right] &\leq c_\phi \mathbb{E} \left[ \phi \left( |M_T^{k,X}| \right) \right] \leq \\ c_\phi \mathbb{E} \left[ \phi \left( \sup_{0 \leq t \leq T} |M_t^{k,X}| \right) \right] &\leq C_\phi \mathbb{E} \left\{ \phi \left( [M^{k,X}, M^{k,X}]_T^{\frac{1}{2}} \right) \right\}, \end{aligned}$$

where  $c_\phi$  and  $C_\phi$  are constants which depends on the Young moderate function  $\phi$ . The uniform integrability of  $\{[M^{k,X}, M^{k,X}]_T^{\frac{1}{2}}; k \geq 1\}$  allows us to state that there exists at least one Young moderate function, also denoted by  $\phi$ , such that

$$\sup_{k \geq 1} \mathbb{E} \left[ \phi \left( \sup_{0 \leq t \leq T} |Z_t^{k,X}| \right) \right] \leq C_\phi \sup_{k \geq 1} \mathbb{E} \left\{ \phi \left( [M^{k,X}, M^{k,X}]_T^{\frac{1}{2}} \right) \right\} < \infty.$$

Therefore  $\{\sup_{0 \leq t \leq T} |Z_t^{k,X}|; k \geq 1\}$  is uniformly integrable and thanks to Th. 1 in [15] we conclude that the set of martingales  $\{Z^{k,X} : k \geq 1\}$  is relatively compact with respect to the weak topology in  $H^1$ .

Reciprocally, let us suppose that the sequence  $\{Z^{k,X} : k \geq 1\}$  is weak relatively compact in  $H^1$ . Again, by applying ([15], Th.1) we know that  $\{\sup_{0 \leq t \leq T} |Z_t^{k,X}|; k \geq 1\}$  is uniformly integrable. By means of Doob and Burkholder inequalities, we then obtain that the family  $\{[M^{k,X}, M^{k,X}]_T^{\frac{1}{2}}; k \geq 1\}$  is uniformly integrable.  $\square$



**Lemma 3.2.** *If  $S$  is an  $\mathbb{F}$  - stopping time then there exists a sequence of positive random variables  $(S_k)_{k \geq 1}$  such that  $S_k$  is an  $\mathbb{F}^k$  - stopping time for each  $k \geq 1$  and  $\lim_{k \rightarrow \infty} \mathbb{P}(S_k = S) = 1$ . Moreover, for any  $G \in \mathcal{F}_S$  there exists a sequence of sets  $(G^k)_{k \geq 1}$  such that  $G^k \in \mathcal{F}_{S_k}^k$ ,  $G^k \subset G \cap \{S < \infty\}$  for every  $k \geq 1$ , and*

$$\lim_{k \rightarrow \infty} \mathbb{P}[G \cap \{S < \infty\} - G^k] = 0.$$

*Proof.* Let  $S$  be an arbitrary  $\mathbb{F}$  - stopping time. Since the graph  $[[S]]$  belongs to  $\mathcal{O}$  we may find a sequence  $(O^k)_{k \geq 1}$  satisfying item (iii) in Lemma 2.2. For an arbitrary  $\varepsilon > 0$ , let  $k$  be large enough in such way that

$$\mathbb{P}[\pi([S]) - \pi(O^k)] < \varepsilon/2.$$

From the standard section theorem there exists an  $\mathbb{F}^k$  - stopping time  $S_k$  such that

$$[[S_k]] \subset O^k \subset [[S]] \text{ and } \mathbb{P}[\pi(O^k)] \leq \mathbb{P}[S_k < \infty] + \varepsilon/2.$$

Then it follows that  $\mathbb{P}[\pi(S) - \pi([S_k])] = \mathbb{P}[S_k \neq S] < \varepsilon$  for  $k$  large enough. This allows us to conclude the first part of the Lemma.

For the second part, let us recall that  $\mathcal{F}_S \cap \{S < \infty\} = \{\Phi^{-1}(O) : O \in \mathcal{O}\}$ , where  $\Phi(w) = (S(w), w)$  for any  $w \in \{S < \infty\}$ . Then, for any  $G \in \mathcal{F}_S$ , there exists an optional set  $J \in \mathcal{O}$  such that  $G \cap \{S < \infty\} = \Phi^{-1}(J)$ . We denote by  $J^k$  the sequence of sets satisfying item (iii) in Lemma 2.2 and  $G^k = \Phi_k^{-1}(J^k)$ , where  $\Phi_k(w) = (S^k(w), w)$  for any  $w \in \{S^k < \infty\}$  and  $(S_k)_{k \geq 1}$  is the sequence of stopping times obtained from the first part. Then, we conclude that  $\mathbb{P}[G \cap \{S < \infty\} - G^k] \leq \mathbb{P}[\pi(J) - \pi(J^k)] \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

Summing up the above lemmas we arrive at the following result.

**Proposition 3.1.** *Assume that  $X$  has finite energy along the filtration family  $(\mathbb{F}^k)_{k \geq 1}$ . Then the set  $\{M^{k,X}; k \geq 1\}$  is  $\sigma(B^1, M^\infty)$  - relatively sequentially compact.*

*Proof.* The Burkholder inequality implies that  $\{\sup_{0 \leq t \leq T} |M_t^{k,X}|; k \geq 1\}$  is uniformly integrable. Moreover, by applying Lemma 3.1 we know that  $\mathcal{Z} = \{Z^{k,X}; k \geq 1\}$  is weakly relatively compact in  $H^1$  and therefore any sequence in  $\mathcal{Z}$  admits a weakly convergent subsequence. With a slight abuse of notation, let us denote by  $Z^{k,X}$  this convergent subsequence in  $H^1$  and  $Z$  the respective  $H^1$ -martingale limit point. Let us fix an  $\mathbb{F}$  - stopping time  $S$  which is bounded by the terminal time  $T$ .

We now claim that  $M^{k,X} \rightarrow Z$  in  $\sigma(B^1, \Lambda^\infty)$ . In other words,

$$\lim_{k \rightarrow \infty} \int M_S^{k,X} g d\mathbb{P} = \int Z_S g d\mathbb{P}$$

holds for every  $g \in L^\infty(\mathcal{F}_S)$ . Recall that (see [34], Corollary 2, pp. 118) it is sufficient to prove for indicator functions  $g = \mathbb{1}_G$  where  $G \in \mathcal{F}_S$ . From Lemma 3.2 there exist a sequence of stopping times  $S_k$  and  $G^k \in \mathcal{F}_{S_k}^k$  satisfying  $\lim_{k \rightarrow \infty} \mathbb{P}[G - G^k] = 0$ . By construction, one should notice from the proof of Lemma 3.2 that  $S_k = S$  on  $G^k$  for every  $k \geq 1$ . Moreover, the martingale property yields

$$\begin{aligned}
\int_G Z_S^{k,X} d\mathbb{P} &= \int_{G-G^k} M_T^{k,X} d\mathbb{P} + \int_{G^k} \mathbb{E} \left[ M_T^{k,X} \mid \mathcal{F}_{S^k}^k \right] d\mathbb{P} \\
&= \int_{G-G^k} M_T^{k,X} d\mathbb{P} + \int_{G^k} M_{S^k}^{k,X} d\mathbb{P} \\
&= \int_{G-G^k} M_T^{k,X} d\mathbb{P} + \int_{G^k} M_S^{k,X} d\mathbb{P} \\
&= \int_{G-G^k} M_T^{k,X} d\mathbb{P} - \int_{G-G^k} M_S^{k,X} d\mathbb{P} \\
&\quad + \int_G M_S^{k,X} d\mathbb{P}.
\end{aligned}$$

Therefore, the uniform integrability assumption yields

$$\left| \int_G Z_S^{k,X} d\mathbb{P} - \int_G M_S^{k,X} d\mathbb{P} \right| = \left| \int_{G-G^k} M_T^{k,X} d\mathbb{P} - \int_{G-G^k} M_S^{k,X} d\mathbb{P} \right| \longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This shows that  $\lim_{k \rightarrow \infty} (M^{k,X}, C) = (Z, C)$  for every  $C \in \Lambda^\infty$  and therefore we may conclude that  $\lim_{k \rightarrow \infty} M^{k,X} = Z$  in the  $\sigma(B^1, \Lambda^\infty)$  - topology. The uniform integrability of  $\{\sup_{0 \leq t \leq T} |M_t^{k,X}|; k \geq 1\}$  and Remark 3.2 allow us to conclude the proof.  $\square$

From the proof of Proposition 3.1 we actually prove the following result.

**Corollary 3.1.** *If  $X$  has finite energy then any sequence in the set  $\{M^{k,X}; k \geq 1\}$  has a subsequence which converges to an  $\mathbb{F}$  - martingale in the  $\sigma(B^1, M^\infty)$  topology. Moreover, all limit points are square-integrable  $\mathbb{F}$ -martingales.*

Now we are interested in studying the full convergence of the canonical decomposition  $\delta^k X = X_0 + M^{k,X} + N^{k,X}$ . For this, a suitable notion of covariation will play a key rule in this work. Since we are primarily concerned with processes which are not semimartingales we must state precisely what we mean by covariation in this context. Our definition of quadratic variation slightly differs from the classical literature in stochastic analysis in a pure space-filtration discretization setup. Moreover, the convergence topology is weaker (in some sense) than usual uniform convergence in probability.

We stress here that it is not our purpose to give a more general definition of a quadratic variation. Instead, we only need a slightly different type of approximation due to the (a priori) lack of regularity of the Wiener functionals.

**Definition 3.2.** *Let  $X$  and  $Y$  be Wiener functionals with  $\mathbb{F}^k$  - projections,  $\delta^k X$  and  $\delta^k Y$ , respectively. We say that  $X$  admits the  $\delta$ -covariation w.r.t  $Y$  if the following limit*

$$(3.5) \quad \langle X, Y \rangle_t^\delta := \lim_{k \rightarrow \infty} [\delta^k X, \delta^k Y]_t$$

*exists weakly in  $L^1$  for every  $t \in [0, T]$ .*

**Remark 3.4.** *Of course, even when*

$$\lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} (X_{T_n^k} - X_{T_{n-1}^k})(Y_{T_n^k} - Y_{T_{n-1}^k}) \mathbb{1}_{\{T_n^k \leq t\}}$$

*exists weakly in  $L^1$  for every  $t \in [0, T]$ , we have in general*

$$\langle X, Y \rangle^\delta \neq \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} (X_{T_n^k} - X_{T_{n-1}^k})(Y_{T_n^k} - Y_{T_{n-1}^k}) \mathbb{1}_{\{T_n^k \leq \cdot\}}.$$

*Moreover, one can check that the  $\delta$ -covariation given in (3.5) does not coincide with the covariations defined via regularization [37]. See however Corollary 3.2 regarding semimartingales. One should notice that the continuity of the pair  $(N^{k,X}, N^{k,Y})$  yields*

$$[\delta^k X, \delta^k Y]_\cdot = [M^{k,X}, M^{k,Y}]_\cdot.$$

Next, we prove some technical results which will allow us to state Theorem 3.1 which is the main result of this section. Not surprisingly, the quadratic variation and energy notions will play a key rule in our result.

**Lemma 3.3.** *Let  $H_t = \mathbb{E}[\mathbb{1}_G | \mathcal{F}_t]$  and  $H_t^k = \mathbb{E}[\mathbb{1}_G | \mathcal{F}_t^k]$  be positive and uniformly integrable martingales with respect to the filtrations  $\mathbb{F}$  and  $\mathbb{F}^k$ , respectively, where  $G \in \mathcal{F}_T$ . If  $W$  is a bounded  $\mathbb{F}$ -martingale then*

$$\left\| \int_0^\cdot H_s dW_s - \oint_0^\cdot H_s^k d\delta^k W_s \right\|_{B^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* Throughout the proof we write  $C$  to denote a positive constant which may differ from line to line. Let us write

$$\begin{aligned} \int_0^t H_s dW_s - \oint_0^t H_s^k d\delta^k W_s &= \int_0^t [H_s - H_s^k] dW_s + \left[ \int_0^t H_s^k dW_s - \oint_0^t H_s^k d\delta^k W_s \right] \\ &=: T_k^1(t) + T_k^2(t) \quad 0 \leq t \leq T. \end{aligned}$$

From the weak convergence of  $\mathbb{F}^k$  to  $\mathbb{F}$  (see (ii) in Lemma 2.2) and the fact that  $H$  is a continuous process it follows that

$$(3.6) \quad H^k \rightarrow H \text{ uniformly in probability as } k \rightarrow \infty.$$

Since

$$\left\| \int_0^\cdot (H_t^k - H_t) dW_t \right\|_{B^2} \leq C \mathbb{E}^{1/2} \sup_{0 \leq t \leq T} |H_t - H_t^k|^2 [W, W]_T,$$

we may conclude that  $T_k^1 \rightarrow 0$  in  $B^2$  as  $k \rightarrow \infty$ .

In order to prove that  $T_k^2$  vanishes when  $k \rightarrow \infty$ , we split it into the following terms

$$(3.7) \quad T_k^2(t) = \int_0^t [H_s^k - H_{s-}^k] dW_s - \oint_0^t [H_s^k - H_{s-}^k] d\delta^k W_s + \int_0^t H_{s-}^k dW_s - \int_0^t H_{s-}^k d\delta^k W_s,$$

where  $\int_0^t H_{s-}^k d\delta^k W_s$  is considered in the Lebesgue-Stieltjes sense. Since  $W$  is bounded it follows that

$$\begin{aligned} \left\| \int_0^\cdot [H_s^k - H_{s-}^k] dW_s \right\|_{\mathbb{B}^2} &\leq C\mathbb{E}^{1/2} \sup_{0 \leq t \leq T} |H_t^k - H_{t-}^k|^2 [W, W]_T \\ &\leq C\mathbb{E}^{1/2p} \sup_{0 \leq t \leq T} |H_t^k - H_{t-}^k|^{2p}, \end{aligned}$$

for  $p > 1$ . The quasi-left continuity of the family  $(\mathbb{F}^k)_{k \geq 1}$ , the uniform integrability of  $\{\sup_{0 \leq t \leq T} |H_t^k - H_{t-}^k|^{2p}; k \geq 1\}$  and (3.6) allow us to conclude from the above estimate that

$$\left\| \int_0^\cdot [H_s^k - H_{s-}^k] dW_s \right\|_{\mathbb{B}^2} \rightarrow 0,$$

as  $k \rightarrow \infty$ . By using the representation  $\delta^k W_t = \oint_0^t \mathcal{D}_s \delta^k W dA_s^k$ , we shall estimate in the same way

$$\begin{aligned} \left\| \oint_0^\cdot [H_s^k - H_{s-}^k] d\delta^k W_s \right\|_{\mathbb{B}^2} &\leq C\mathbb{E}^{1/2} \int_0^T |H_s^k - H_{s-}^k|^2 d[\delta^k W, \delta^k W]_s \\ &\leq C\mathbb{E}^{1/2} \sup_{0 \leq t \leq T} |H_t^k - H_{t-}^k|^2 [\delta^k W, \delta^k W]_T \\ &\leq C\mathbb{E}^{1/2p} \sup_{0 \leq t \leq T} |H_t^k - H_{t-}^k|^{2p} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ . It remains to estimate the last part in (3.7). From the weak convergence of  $\mathbb{F}^k$  to  $\mathbb{F}$  and (2.11), one should notice that  $L^k = W - \delta^k W$  is an  $\mathbb{F}$ -semimartingale such that  $L^k \rightarrow 0$  uniformly in probability as  $k \rightarrow \infty$ . We shall write

$$\int_0^t H_{s-}^k dW_s - \int_0^t H_{s-}^k d\delta^k W_s = \int_0^t H_{s-}^k dL_s^k.$$

We now claim that  $(H^k, L^k)$  satisfies the assumptions of Theorem 2.2 in [31]. At first, by the linearity of the conditional expectation operator we do have  $(H^k, L^k) \rightarrow (H, 0)$  in probability on the two-dimensional Skorohod space. Moreover, one can easily check that  $L^k$  satisfies assumption C2.2(i) in [31], and therefore

$$\int_0^\cdot H_{s-}^k dL_s^k \rightarrow 0 \quad \text{uniformly in probability.}$$

Again by usual compactness arguments we do have convergence in  $\mathbb{B}^2$ . □

In the sequel, if  $X$  is a Wiener functional and  $S$  is a stopping time then we denote  $X_t^S := X_{S \wedge t}$ .

**Lemma 3.4.** *Let  $X$  be a finite energy Wiener functional with the  $\mathbb{F}^k$ -decomposition given by  $(M^{k,X}, N^{k,X})$ . Let  $M^{k,X}$  be a convergent subsequence such that  $M^{k,X} \rightarrow Z$  in  $\sigma(B^1, M^\infty)$ , where  $Z \in H^2$ . If  $W \in H^2$ , then*

$$\lim_{k \rightarrow \infty} [M^{k,X}, \delta^k W]_t = [Z, W]_t \quad \text{weakly in } L^1$$

for every  $t \in [0, T]$ .

*Proof.* With a slight abuse of notation, let  $Z^{k,X}$  be the  $\mathbb{F}$ -martingale subsequence obtained from (3.4), Proposition 3.1 and Lemma 3.1 such that  $\lim_{k \rightarrow \infty} Z^{k,X} = Z$  and  $\lim_{k \rightarrow \infty} M^{k,X} = Z$  in  $\sigma(B^1, M^\infty)$ . Thanks to Th.7 [15], we know that  $[Z^{k,X}, U]_t \rightarrow [Z, U]_t$  weakly in  $L^1(\mathbb{P})$  for every  $t \in [0, T]$  and  $U$  a BMO  $\mathbb{F}$ -martingale. Given  $G \in \mathcal{F}_T$ , let us consider the  $\mathbb{F}^k$ -martingale  $H_t^k = \mathbb{E}[\mathbb{1}_G | \mathcal{F}_t^k]$ . Let  $W$  be a bounded  $\mathbb{F}$ -martingale. At first, one should notice that the finite energy assumption gives  $M^{k,X} \in H^2(\mathbb{F}^k)$  for every  $k \geq 1$ . By using the  $\mathbb{F}^k$ -dual optional projection property we shall write

$$\begin{aligned} \mathbb{E}[\mathbb{1}_G[M^{k,X}, \delta^k W]_t] &= \mathbb{E}\left[\int_0^t H_s^k d[M^{k,X}, \delta^k W]_s\right] \\ &= \mathbb{E}[M^{k,X}, J^k]_t = \mathbb{E}[M_t^{k,X} J_t^k] \quad 0 \leq t \leq T, \end{aligned}$$

where  $J^k$  is the  $\mathbb{F}^k$ -square-integrable martingale given by the optional integral  $\oint H^k d\delta^k W$ . In the same manner, we have that

$$\begin{aligned} \mathbb{E}[\mathbb{1}_G[Z^{k,X}, W]_t] &= \mathbb{E}\left[\int_0^t H_s d[Z^{k,X}, W]_s\right] \\ &= \mathbb{E}[Z^{k,X}, J]_t = \mathbb{E}[Z_t^{k,X} J_t] \quad 0 \leq t \leq T, \end{aligned}$$

where  $J$  is the stochastic integral  $\int H dW$  and  $H = \mathbb{E}[\mathbb{1}_G | \mathcal{F}]$ . Moreover,

$$\begin{aligned} \mathbb{E}[Z_t^{k,X} J_t] - \mathbb{E}[M_t^{k,X} J_t^k] &= \mathbb{E}[M_t^{k,X} (J_t - J_t^k)] - \mathbb{E}[(M_t^{k,X} - Z_t^{k,X}) J_t] \\ &=: T_1^k(t) + T_2^k(t) \quad 0 \leq t \leq T. \end{aligned}$$

We fix  $t \in [0, T]$  and we notice that it is sufficient to prove that  $T_1^k(t) + T_2^k(t) \rightarrow 0$  as  $k \rightarrow \infty$ . The first term  $\lim_{k \rightarrow \infty} T_1^k(t) = 0$  because of the finite energy assumption and Lemma 3.3. For the second term, let  $S_N$  be a localizing sequence of stopping times in such way that  $J^{S_N}$  is a bounded martingale for every  $N \geq 1$ . The second term can be estimated as follows

$$\begin{aligned} |T_2^k(t)| &\leq \mathbb{E}^{1/2} |M_t^{k,X} - Z_t^{k,X}|^2 \mathbb{E}^{1/2} |J_t - J_t^{S_N}|^2 + |\mathbb{E}(M_t^{k,X} - Z_t^{k,X}) J_t^{S_N}| \\ &\leq C \mathbb{E}^{1/2} |J_t - J_t^{S_N}|^2 + |\mathbb{E}(M_t^{k,X} - Z_t^{k,X}) J_t^{S_N}|. \end{aligned}$$

By noting that both subsequences  $Z^{k,X}$  and  $M^{k,X}$  converge to the same limit in  $\sigma(B^1, M^\infty)$ , the above estimate allows us to conclude the proof of the Lemma for bounded martingales and since they are dense in  $H^2$ , the result follows.  $\square$

The following immediate consequence of Lemma 3.4 gives the existence of the  $\delta$ -covariation  $\langle X, Y \rangle^\delta$  for any  $X$  and  $Y$  square-integrable continuous  $\mathbb{F}$ -semimartingales.

**Corollary 3.2.** *If  $X$  and  $Y$  are square-integrable continuous  $\mathbb{F}$ -semimartingales then  $\langle X, Y \rangle^\delta = [X, Y]$ .*

*Proof.* At first, we consider a pair of martingales  $(M, W)$ . From (2.11) we know that  $\delta^k M_t = \mathbb{E}[M_T | \mathcal{F}_t^k]$  for every  $M \in \mathcal{H}^2$ . The weak convergence of  $\mathbb{F}^k$  to  $\mathbb{F}$  (see (ii) Lemma 2.2) yields  $\delta^k M \rightarrow M$  uniformly in probability and since  $\mathcal{E}_2(M) < \infty$  we do have  $\delta^k M \rightarrow M$  strongly in  $B^1$ . A simple application of Lemma 3.4 then gives

$$\langle M, W \rangle_t^\delta = \lim_{k \rightarrow \infty} [\delta^k M, \delta^k W]_t = [M, W]_t,$$

where the limit holds weakly in  $L^1$  for every  $0 \leq t \leq T$ .

It remains to check that

$$\langle X, Y \rangle^\delta = 0 \quad \text{if } X \text{ and } Y \text{ are bounded variation processes,}$$

and

$$\langle X, Y \rangle^\delta = 0 \quad \text{if } Y \in \mathcal{H}^2 \text{ and } X \text{ is a bounded variation processes.}$$

Let us fix  $g \in L^\infty$ ,  $t \in [0, T]$  and  $X$  and  $Y$  two square integrable  $\mathbb{F}$ -semimartingales. Let  $H^k = \mathbb{E}[g | \mathcal{F}^k]$ . The  $\mathbb{F}^k$ -optional duality and Kunita-Watanabe inequality yield

$$\begin{aligned} |\mathbb{E}g[\delta^k X, \delta^k Y]_t| &= \left| \mathbb{E} \int_0^t H_s^k d[\delta^k X, \delta^k Y]_s \right| \\ &\leq \mathbb{E} \int_0^T |H_s^k| |d[\delta^k X, \delta^k Y]_s| \\ (3.8) \quad &\leq C \mathbb{E}^{1/2}[\delta^k X, \delta^k X]_T \times \mathbb{E}^{1/2}[\delta^k Y, \delta^k Y]_T. \end{aligned}$$

If  $X$  and  $Y$  are bounded variation processes then  $\mathcal{E}_2(X) = \mathcal{E}_2(Y) = 0$ . If  $X \in \mathcal{H}^2$  and  $Y$  has bounded variation, we have  $\mathcal{E}_2(X) < \infty$  and  $\mathcal{E}_2(Y) = 0$ . Therefore, the estimate (3.8) allows us to conclude the proof.  $\square$

Now we are able to prove the main result of this section.

**Theorem 3.1.** *Let  $X$  be a finite energy Wiener functional such that  $\lim_{k \rightarrow \infty} \delta^k X = X$  weakly in  $B^1$  and let  $(M^{k,X}, N^{k,X})$  be the canonical decomposition of  $\delta^k X$ . If  $\langle X, B \rangle^\delta$  exists then there exists a unique martingale  $M^X$  in  $\mathcal{H}^2$  such that  $N^X := X - X_0 - M^X$  satisfies the following orthogonality condition*

$$\langle N^X, W \rangle^\delta \equiv 0,$$

for every  $W \in \mathcal{H}^2$ . If this is the case, we may write  $X = X_0 + M^X + N^X$  and this decomposition is unique. Moreover,  $M^{k,X} \rightarrow M^X$  and  $N^{k,X} \rightarrow N^X$  weakly in  $B^1$ .

*Proof.* If  $X$  has finite energy then by Proposition 3.1 we know that  $\{M^{k,X}; k \geq 1\}$  is  $\sigma(B^1, M^\infty)$ -relatively sequentially compact. By assumption  $\langle X, B \rangle^\delta$  exists and therefore Lemma 3.4 implies that for any martingale limit points  $M$  and  $M'$  we have  $[M - M', B] = 0$ . In this case,  $M^{k,X}$  should be convergent and  $\langle X, B \rangle^\delta = [M^X, B]$ , where  $M^X = \lim_{k \rightarrow \infty} M^{k,X}$ . Let  $N^X := \lim_{k \rightarrow \infty} N^{k,X}$  weakly in  $B^1$ . Let  $W$  be an  $\mathbb{F}$ -square-integrable martingale. By the very definition

$$\delta^k N^X = M^{k,X} - \delta^k M^X + N^{k,X},$$

and the continuity of  $N^{k,X}$  yields

$$[\delta^k N^X, \delta^k W]_t = [M^{k,X} - \delta^k M^X, \delta^k W]_t, \quad 0 \leq t \leq T; k \geq 1.$$

Corollary 3.2 and Lemma 3.4 allow us to conclude that  $\langle N^X, W \rangle^\delta = 0$ . The uniqueness of the decomposition is an immediate consequence of the orthogonality of the non-martingale part combined with Corollary 3.2.  $\square$

**Remark 3.5.** *Theorem 3.1 is very reminiscent from the early work of Graversen and Rao [24] where the concept of energy plays a key rule. More recently, Coquet et al. [11] have shown the connection between energy, weak Dirichlet processes and existence of covariation. Up to a quadratic variation notion, the result given in Theorem 3.1 is analogous to Theorem 2.1 in [11]. There are two fundamental differences here: Firstly, we are able to show explicitly a robust approximating sequence of semimartingales in terms of a stochastic derivative. See Section 5. Secondly, the covariation operation  $\langle \cdot, \cdot \rangle^\delta$  is rather different from previous works on Dirichlet processes (see e.g [6], [19], [13], [10] and other references therein), although in some cases may coincide (see e.g. Corollary 3.2 and Lemma 7.4). The key point here is that the relevant structure in our decomposition comes only from  $(T_n^k)_{k,n \geq 1}$  via the embedded semimartingale skeleton  $\delta^k X$ .*

It is natural to ask what relation the approximation stated in Theorem 3.1 has to standard decompositions. The conditions in Theorem 3.1 do not appear to be easily verifiable at first sight. The next section is devoted to providing useful characterizations which provide sufficient conditions for the assumptions in Theorem 3.1 to hold. Moreover, we are going to prove that under a mild path regularity, the finite energy property is sufficient in order to get a robust pre-limit semimartingale sequence.

#### 4. THE $\delta$ -WEAK DIRICHLET PROCESSES AND OTHER DECOMPOSITIONS

Throughout this section, all the Wiener functionals are assumed to belong to  $B^1$ . This is not a strong assumption since by localization all the Wiener functionals are locally bounded. In the sequel, we give some useful characterizations of Theorem 3.1. The first result states that under strong  $B^1$ -convergence the following result holds.

**Proposition 4.1.** *If a Wiener functional  $X$  satisfies  $\delta^k X \rightarrow X$  strongly in  $B^1$  and  $\mathcal{E}_2(X) < \infty$  then  $\langle X, B \rangle^\delta$  exists and it satisfies the assumptions in Theorem 3.1*

*Proof.* Throughout this proof  $C$  will denote a generic constant which may differ from line to line. We claim that under the above conditions  $\lim_{k \rightarrow \infty} [\delta^k X, A^k]_t$  exists weakly in  $L^1$  for every  $0 \leq t \leq T$ . For this, we fix  $t \in [0, T]$ ,  $g \in L^\infty$  and we will show that

$$\mathbb{E}g[\delta^k X, A^k]_t \text{ is Cauchy in } \mathbb{R}.$$

We have

$$\begin{aligned}
|\mathbb{E}g([\delta^k X, A^k]_t - [\delta^{k-1} X, A^{k-1}]_t)| &\leq |\mathbb{E}g[\delta^k X, A^k - A^{k-1}]_t| \\
&+ |\mathbb{E}g[\delta^k X - \delta^{k-1} X, A^{k-1}]_t| \\
&=: J_1^k(t) + J_2^k(t)
\end{aligned}$$

The first term can be estimated as follows

$$\begin{aligned}
|J_1^k(t)| &\leq C\mathbb{E}[\delta^k X, \delta^k X]_T^{1/2} [A^k - A^{k-1}, A^k - A^{k-1}]_T^{1/2} \\
&\leq C\mathcal{E}_2^{1/2}(X)\mathbb{E}^{1/2}[A^k - A^{k-1}, A^k - A^{k-1}]_T \rightarrow 0
\end{aligned}$$

as  $k \rightarrow \infty$ . For the second term, we recall that  $(T_i^k)_{i \geq 0}$  is a refinement of  $(T_j^{k-1})_{j \geq 0}$  in the sense that for a given  $n \geq 1$  there exists  $j_1$  and  $j_2$  such that  $T_n^{k-1} = T_{j_2}^k$  and  $T_{n-1}^{k-1} = T_{j_1}^k$ . Moreover, by the very definition

$$(4.1) \quad \{(t, \omega); \Delta \delta^{k-1} X_t(\omega) \neq 0\} \subset \bigcup_{n \geq 0} [[T_n^k, T_n^k]], \quad \{(t, \omega); \Delta A_t^{k-1}(\omega) \neq 0\} \subset \bigcup_{n \geq 0} [[T_n^k, T_n^k]].$$

If  $Y_t^k := \delta^k X_t - \delta^{k-1} X_t$  we notice that

$$\begin{aligned}
\Delta Y_{T_n^k}^k &= \Delta \delta^k X_{T_n^k} - (\delta^{k-1} X_{T_n^k} - \delta^{k-1} X_{T_{n-1}^k}) \\
&= \Delta \delta^k X_{T_n^k} - (\delta^{k-1} X_{T_n^k} - \delta^{k-1} X_{T_{n-1}^k}) \\
(4.2) \quad &= (\delta^k X_{T_n^k} - \delta^{k-1} X_{T_n^k}) + (\delta^{k-1} X_{T_{n-1}^k} - \delta^k X_{T_{n-1}^k}) \quad \text{a.s on } \{\Delta A_{T_{n-1}^k}^k \neq 0\}.
\end{aligned}$$

Taking into account (4.1) and (4.2), we arrive at the following estimate

$$\begin{aligned}
|J_2^k(t)| &\leq 2^{-(k-1)} \left| \sum_{n \geq 1} \int_{\{\Delta A_{T_n^k}^{k-1} > 0, T_n^k \leq t\}} g \Delta Y_{T_n^k}^k d\mathbb{P} \right| \\
&+ 2^{-(k-1)} \left| \sum_{n \geq 1} \int_{\{\Delta A_{T_n^k}^{k-1} < 0, T_n^k \leq t\}} g \Delta Y_{T_n^k}^k d\mathbb{P} \right| \\
&=: J_{2,1}^k(t) + J_{2,2}^k(t).
\end{aligned}$$

It is sufficient to estimate  $J_{2,1}^k(t)$  since the other term can be estimated in the same way. We now claim that  $J_{2,1}^k(t) = \{J_{2,1}^k(t); k \geq 1\}$  is relatively compact in  $\mathbb{R}$  where all limit points are equal to zero. With a slight abuse of notation, let  $(J_{2,1}^k(t))_{k \geq 1}$  be a sequence in  $J_{2,1}^k$ . By the strong convergence  $\delta^k X \rightarrow X$  in  $B^1$ , for a given  $m \geq 1$ , we may pick up  $k_0(m)$  such that



$$(4.3) \quad \mathbb{E} \sup_{0 \leq t \leq T} |\delta^k X_t - \delta^{k-1} X_t| < \frac{1}{2^m} \quad \forall k \geq k_0(m).$$

Therefore, from (4.2) and (4.3) it follows that

$$\left| \sum_{n=1}^m \int_{\{\Delta A_{T_n^k}^{k-1} > 0, T_n^k \leq t\}} g \Delta Y_{T_n^k}^k d\mathbb{P} \right| \leq C \sum_{n=1}^m \frac{1}{2^n}$$

for  $k$  sufficiently large. In this way, we may find a subsequence such that

$$\lim_{i \rightarrow \infty} \left| \sum_{n=1}^{\infty} \int_{\{\Delta A_{T_n^{k(i)}}^{k(i)-1} > 0, T_n^{k(i)} \leq t\}} g \Delta Y_{T_n^{k(i)}}^{k(i)} d\mathbb{P} \right| \leq C \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

This shows that  $\lim_{i \rightarrow \infty} J_{2,1}^{k(i)}(t) = 0$ . Moreover, from the above argument we notice that any sequence in  $J_{2,1}^k$  admits a subsequence which converges to zero and therefore the entire sequence  $J_{2,1}^k$  must converge to zero. Hence, we may conclude that  $\langle X, B \rangle_t^\delta$  exists for every  $t \in [0, T]$  and therefore  $X$  will satisfy the assumptions of Theorem 3.1.  $\square$

We recall that our concept of finite energy (Definition 3.1) is taken with a conditional expectation point of view. It is natural to ask what happens without conditioning on the information flow  $\mathcal{G}_n^k$ ,  $k, n \geq 1$ . In the sequel, we introduce the following notion of energy

$$\mathcal{E}_2^s(X) := \sup_{k \geq 1} \mathbb{E} \sum_{n \geq 1} (X_{T_n^k} - X_{T_{n-1}^k})^2 \mathbb{1}_{\{T_n^k \leq T\}}.$$

The quantity  $\mathcal{E}_2^s(X)$  coincides with the classical notion of energy restricted on the family of stopping times  $\{(T_n^k)_{n \geq 0}; k \geq 1\}$  and it is certainly easier to check than  $\mathcal{E}_2(X)$ . We do not know if in general  $\mathcal{E}_2^s(X) \neq \mathcal{E}_2(X)$  but the next result gives a useful result in this direction. If  $X$  is a Wiener functional then we shall define the following process

$$\mathcal{H}^{k,X} := \sum_{n=1}^{\infty} \frac{X_{T_n^k} - X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} \mathbb{1}_{[[T_n^k, T_n^k]]}.$$

**Proposition 4.2.** *The  $\mathbb{F}^k$ -stochastic derivative  $\mathcal{D}\delta^k X$  given in (2.13) is the  $\mathbb{F}^k$ -optional projection of  $\mathcal{H}^{k,X}$ . Moreover,*

$$(4.4) \quad \mathcal{E}_2(X) \leq \mathcal{E}_2^s(X).$$

*Proof.* By the very definition

$$\mathbb{E} \left[ \frac{X_{T_n^k} - X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} \middle| \mathcal{G}_n^k \right] = \frac{\mathbb{E}[X_{T_n^k} | \mathcal{G}_n^k] - \mathbb{E}[X_{T_{n-1}^k} | \mathcal{G}_n^k]}{B_{T_n^k} - B_{T_{n-1}^k}}.$$

We recall that  $\mathcal{G}_n^k = \mathcal{G}_{n-1}^k \vee \sigma(T_n^k, B_{T_n^k} - B_{T_{n-1}^k})$ . Since  $X$  is  $\mathbb{F}$ -adapted we do have

$$X_{T_{n-1}^k} = \mathbb{E}[X_{T_{n-1}^k} | \mathcal{F}_{T_{n-1}^k} \vee \sigma(T_n^k, B_{T_n^k} - B_{T_{n-1}^k})].$$

Therefore, one can easily check that

$$\frac{\mathbb{E}[X_{T_n^k} | \mathcal{G}_n^k] - \mathbb{E}[X_{T_{n-1}^k} | \mathcal{G}_{n-1}^k]}{B_{T_n^k} - B_{T_{n-1}^k}} = \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} = \mathcal{D}_{T_n^k}^k \delta^k X.$$

Since  $\mathcal{F}_S^k \cap \{T_n^k \leq S < T_{n+1}^k\} = \mathcal{G}_n^k \cap \{T_n^k \leq S < T_{n+1}^k\}$  up to null  $\mathbb{P}$ -sets for every  $\mathbb{F}^k$ -stopping time  $S$ , then the representation follows. Now a simple application of Jensen inequality and the  $\mathbb{F}^k$ -optional duality yields

$$\begin{aligned} \mathcal{E}_2(X) &= \sup_{k \geq 1} \mathbb{E} \int_0^T (\mathcal{D}_s \delta^k X)^2 d[A^k, A^k]_s \\ &\leq \sup_{k \geq 1} \mathbb{E} \int_0^T {}^{o,k} \left[ \left( \mathcal{H}_s^{k,X} \right)^2 \right] d[A^k, A^k]_s \\ &= \sup_{k \geq 1} \mathbb{E} \int_0^T (\mathcal{H}_s^{k,X})^2 d[A^k, A^k]_s \\ &= \sup_{k \geq 1} \mathbb{E} \left[ \sum_{n=1}^{\infty} \left( X_{T_n^k} - X_{T_{n-1}^k} \right)^2 \mathbb{1}_{\{T_n^k \leq T\}} \right] = \mathcal{E}_2^s(X). \end{aligned}$$

□

Since all results in this paper holds for  $\mathcal{E}_2$  we continue our analysis along this energy concept. In the sequel, it will be useful to introduce the following notations. We write  $X \perp H^2$  to denote that  $\langle X, W \rangle^\delta = 0$  for every  $W \in H^2$ . In view of Theorem 3.1, it is natural to introduce a class of finite energy Wiener functionals  $X$  which can be decomposed as

$$(4.5) \quad X = X_0 + M + V,$$

where  $M \in H^2$  and  $V \perp H^2$ . Such decomposition will be called a  $\delta$ -decomposition. Bearing in mind the concept of weak Dirichlet processes (see e.g [19, 11, 13] and other references therein), the following definition is natural.

**Definition 4.1.** *The class of Wiener functionals which admits a decomposition of the form (4.5) are called finite energy  $\delta$ -weak Dirichlet processes.*

**Remark 4.1.** *By the orthogonality relation and the fact that  $\langle M, W \rangle^\delta = [M, W]$  for every  $M, W \in H^2$ , if there exists a  $\delta$ -decomposition then it must be unique.*

Let us begin with the localization results. The following definition is natural.

**Definition 4.2.** *We say that  $X$  is a local finite energy  $\delta$ -weak Dirichlet process if there exists a non-decreasing sequence of  $\mathbb{F}$ -stopping times  $(T_m)_{m \geq 0}$  with  $\lim_{m \rightarrow \infty} T_m = \infty$  a.s such that, for each  $m$ , the stopped process  $X^{T_m}$  is a finite energy  $\delta$ -weak Dirichlet process.*

Next we aim at providing local properties for the class of finite energy  $\delta$ -weak Dirichlet process. For this, we need the following lemmas.

**Lemma 4.1.** *If  $M \in H^2$  then  $[\delta^k M, A^k]_S \rightarrow [M, B]_S$  weakly in  $L^1$  for any  $\mathbb{F}$ -stopping time  $S$  bounded by  $0 < T < \infty$ .*

*Proof.* Let  $\mathcal{S}$  be the algebra generated by the sets  $G \times \{0\}$  and  $G \times (s, t]$  where  $G \in \mathcal{F}_T$  and  $0 \leq s < t \leq T$ . A direct application of Corollary 3.2 yields

$$(4.6) \quad \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \mathbb{1}_{\{D\}} d[\delta^k M, A^k]_s = \mathbb{E} \int_0^T \mathbb{1}_{\{D\}} d[M, B]_s,$$

for any  $D \in \mathcal{S}$ . We claim that  $\mathcal{S}$  is a monotone class. Let  $(D_i)_{i \geq 1}$  be a sequence in  $\mathcal{S}$  such that  $D_i \subset D_{i+1}$  for any  $i \geq 1$  and let  $D = \cup_{i=1}^{\infty} D_i$ . We shall write

$$\begin{aligned} & \mathbb{E} \int_0^T \mathbb{1}_{\{D\}}(s) d[\delta^k M, A^k]_s - \mathbb{E} \int_0^T \mathbb{1}_{\{D\}}(s) d[M, B]_s = \\ & \mathbb{E} \int_0^T \mathbb{1}_{\{D-D_i\}}(s) d[\delta^k M, A^k]_s - \mathbb{E} \int_0^T \mathbb{1}_{\{D-D_i\}}(s) d[M, B]_s + \\ & \mathbb{E} \int_0^T \mathbb{1}_{\{D_i\}}(s) d[\delta^k M, A^k]_s - \mathbb{E} \int_0^T \mathbb{1}_{\{D_i\}}(s) d[M, B]_s =: T_1(k, i) - T_2(i) + T_3(k, i). \end{aligned}$$

By using the finite energy of  $M$  and the fact that  $\sup_{k \geq 1} \sup_{0 \leq t \leq T} |h^k(t)| < \infty$ , we may find a positive constant  $C > 0$  which does not depend on  $k$  and  $i$  such that

$$(4.7) \quad |T_1(k, i)| \leq C \left( \int_0^T \mathbb{P}(D - D_i)_s ds \right)^{1/2},$$

where  $(D - D_i)_s$  is the  $s$ -section of the set  $D - D_i$  for  $0 \leq s \leq T$ . By combining (4.6) and (4.7) we conclude that  $\mathcal{S}$  is a monotone class and therefore it contains the  $\mathcal{F}_T \times \mathcal{B}([0, T])$ . By taking  $[[0, S]] \in \mathcal{F}_T \times \mathcal{B}([0, T])$ , we have

$$\begin{aligned} \lim_{k \rightarrow \infty} [\delta^k M, A^k]_S &= \lim_{k \rightarrow \infty} \int_0^T \mathbb{1}_{[[0, S]]} d[\delta^k M, A^k]_s \\ &= \int_0^T \mathbb{1}_{[[0, S]]} d[M, B]_s = [M, B]_S \quad \text{weakly in } L^1, \end{aligned}$$

and therefore we may conclude the proof.  $\square$

**Lemma 4.2.** *Let  $X$  be a finite energy Wiener functional such that  $\langle X, B \rangle^\delta = 0$  and let  $S$  be an  $\mathbb{F}$ -stopping time such that  $X^S$  is bounded. Then  $\langle X^S, B \rangle^\delta = 0$  and  $X^S$  has finite energy as well.*

*Proof.* For a given  $t \in [0, T]$ , let us decompose  $[\delta^k X^S, A^k]_t$  as follows

$$\begin{aligned} [\delta^k X^S, A^k]_t &= \mathbb{1}_{\{S \leq t\}} \sum_{n=1}^{\infty} \Delta \delta^k X_{T_n^k} \Delta A_{T_n^k}^k \mathbb{1}_{\{T_n^k \leq S\}} + \mathbb{1}_{\{S > t\}} \sum_{n=1}^{\infty} \Delta \delta^k X_{T_n^k} \Delta A_{T_n^k}^k \mathbb{1}_{\{T_n^k \leq t\}} \\ &+ \sum_{n=1}^{\infty} \left[ \mathbb{E} \left( X_S \mid \mathcal{F}_{T_n^k}^k \right) - \mathbb{E} \left( X_S \mid \mathcal{F}_{T_{n-1}^k}^k \right) \right] \Delta A_{T_n^k}^k \mathbb{1}_{\{S \leq T_{n-1}^k \leq t\}} \\ &+ \sum_{n=1}^{\infty} \left[ \mathbb{E} \left( X_S \mid \mathcal{F}_{T_n^k}^k \right) - \mathbb{E} \left( X_{T_{n-1}^k} \mid \mathcal{F}_{T_{n-1}^k}^k \right) \right] \Delta A_{T_n^k}^k \mathbb{1}_{\{T_{n-1}^k < S < T_n^k \leq t\}} \\ &=: J_1^k + J_2^k + J_3^k + J_4^k. \end{aligned}$$

By using the finite energy and the orthogonality assumptions we can apply the same arguments of Lemma 4.1 to show that  $J_1^k \rightarrow 0$  weakly in  $L^1$  as  $k \rightarrow \infty$ . Again, the orthogonality assumption yields  $J_2^k \rightarrow 0$  weakly in  $L^1$ . The term  $J_4^k$  vanishes weakly because  $|\Delta A_{T_n}^k| = 2^{-k}$  for every  $k, n$  and  $X^S$  is bounded. So it remains to estimate the third term. For this, let us denote  $M. = E(X_S \mid \mathcal{F}.)$  and notice that

$$J_3^k = [\delta^k M, A^k]_t \mathbb{1}_{\{S \leq t\}} - [\delta^k M, A^k]_S \mathbb{1}_{\{S \leq t\}}.$$

Lemma 4.1 yields

$$\begin{aligned} \lim_{k \rightarrow \infty} J_3^k &= \lim_{k \rightarrow \infty} [\delta^k M, A^k]_t \mathbb{1}_{\{S \leq t\}} - \lim_{k \rightarrow \infty} [\delta^k M, A^k]_S \mathbb{1}_{\{S \leq t\}} \\ &= [M, B]_t \mathbb{1}_{\{S \leq t\}} - [M, B]_S \mathbb{1}_{\{S \leq t\}} \\ &= [M, B]_S \mathbb{1}_{\{S \leq t\}} - [M, B]_S \mathbb{1}_{\{S \leq t\}} = 0 \quad \text{weakly in } L^1. \end{aligned}$$

By using the same above arguments, one can easily check that  $X^S$  has finite energy as well.  $\square$

**Definition 4.3.** We say that a given Wiener functional  $V$  is  $\delta$ -orthogonal to  $H^2$  with finite energy if  $V \perp H^2$  and  $\mathcal{E}_2(V) < \infty$ . We say that a given Wiener functional  $V$  is  $\delta$ -locally orthogonal to  $H^2$  with finite energy if there exists a non-decreasing sequence  $(T_m)_{m \geq 0}$  of  $\mathbb{F}$ -stopping times with  $\lim_{m \rightarrow \infty} T_m = \infty$  a.s, such that  $V^{T_m}$  is  $\delta$ -orthogonal to  $H^2$  with finite energy.

The following result is an immediate consequence of the definitions and Lemma 4.2. For the convenience of the reader, we give the details here.

**Proposition 4.3.** If  $X$  is a local finite energy  $\delta$ -weak Dirichlet process then there exists a unique decomposition

$$X = X_0 + M + V$$

where  $M \in H_{loc}^2$  and  $V$  is  $\delta$ -locally orthogonal to  $H^2$  with finite energy.

*Proof.* Let  $(T_n)$  be a localizing sequence of  $X$ . Let

$$X^{T_n} = X_0 + M_t^n + V_t^n$$

be the  $\delta$ -decomposition where  $V^n \perp H^2$  and  $M^n \in H^2$ . We fix  $n \geq 1$  and we consider the stopping times

$$R_m := \inf\{t \geq 0; |V_t^n| > m\} \wedge \inf\{t \geq 0; |V_t^{n+1}| > m\} \wedge T_n \wedge T; \quad m \geq 1.$$

Then

$$\begin{aligned} (M^{n+1})^{R_m} - (M^n)^{R_m} &= (X^{T_{n+1}})^{R_m} - (X^{T_n})^{R_m} \\ &+ (V^n)^{R_m} - (V^{n+1})^{R_m} \\ &= (V^n)^{R_m} - (V^{n+1})^{R_m}. \end{aligned}$$

Lemma 4.2 and Corollary 3.2 give us

$$\langle (M^n - M^{n+1})^{R_m}, B \rangle^\delta = [(M^n - M^{n+1})^{R_m}, B] = 0,$$

and in this case we must have  $(M^n)^{R_m} = (M^{n+1})^{R_m}$  and  $(V^n)^{R_m} = (V^{n+1})^{R_m}$  on  $[[0, R_m[[$  for every  $m \geq 1$ . Moreover, since  $\lim_{m \rightarrow \infty} R_m = T_n \wedge T$  a.s we see that  $M^n$  and  $M^{n+1}$  as well as  $V^n$  and  $V^{n+1}$  coincide on  $[[0, T_n \wedge T[[$ . Therefore, we shall define

$$M := M^n \quad \text{on } [[0, T_n \wedge T[[$$

$$V := V^n \quad \text{on } [[0, T_n \wedge T[[.$$

Of course,  $M \in H_{loc}^2$  and  $V^{T_n \wedge T} \perp H^2$  for every  $n \geq 1$ . We now check that this indeed is the unique decomposition. Let us assume that there is another decomposition  $X = X_0 + Y + Z$ . Let  $I_n$  and  $J_n$  be localizing sequences for  $Z$  and  $Y$ , respectively. That is,  $Y^{J_n} \in H^2$  and  $Z^{I_n} \perp H^2$ ,  $\mathcal{E}_2(Z^{I_n}) < \infty$ . Let  $K_n = \inf\{t > 0; |Z_t| > n\}$ ,  $Q_n = \inf\{t > 0; |V_t| > n\}$  and  $S_n = T_n \wedge I_n \wedge J_n \wedge K_n \wedge Q_n \wedge T$ . Again Lemma 4.2 and Corollary 3.2 give us  $(M^{S_n} - Y^{S_n}) = (Z^{S_n} - V^{S_n})$ , where

$$\langle M^{S_n} - Y^{S_n}, B \rangle^\delta = [M^{S_n} - Y^{S_n}, B] = 0.$$

Since  $M^{S_n} - Y^{S_n}$  is a martingale such that  $M_0^{S_n} - Y_0^{S_n} = 0$  a.s, we do have  $M^{S_n} = Y^{S_n}$  and  $Z^{S_n} = V^{S_n}$  a.s for every  $n \geq 1$ . By taking  $n \rightarrow \infty$  we conclude the proof.  $\square$

Thanks to [[10]; Th.1], we know that if a given Wiener functional  $X$  has continuous paths then

$$(4.8) \quad \mathbb{E}[X, \mathcal{F}^k] \rightarrow X. \quad \text{uniformly in probability}$$

as  $k \rightarrow \infty$ . In view of Proposition 4.1 and (4.8), it is natural to introduce the following property. We say that  $X$  satisfies condition **(BUI)** if

**(BUI):** The family  $\sup_{n \geq 1} \mathbb{E}[X_{T_n^k} | \mathcal{G}_n^k] \mathbb{1}_{\{T_n^k \leq T\}}$  is uniformly integrable.

**Remark 4.2.** By the Doob maximal inequality, one should notice that if  $X \in B^p$  for some  $1 < p < \infty$  then it satisfies condition (BUI).

The combination of condition (BUI), path continuity and finite energy yields the main result of this section.

**Theorem 4.1.** *If  $X$  is a continuous Wiener functional such that  $\mathcal{E}_2(X) < \infty$  (locally) then it is a local finite energy  $\delta$ -weak Dirichlet process and therefore there exists a unique decomposition*

$$(4.9) \quad X = X_0 + M + V,$$

where  $M \in H_{loc}^2$  and  $V$  is  $\delta$ -locally orthogonal to  $H^2$  with finite energy.

*Proof.* Let  $(T_m)_{m \geq 0}$  and  $(R_m)$  be the localizing sequences of stopping times such that  $X^{T_m}$  is bounded and  $\mathcal{E}_2(X^{R_m}) < \infty$  for each  $m \geq 1$ . Let us define  $S_m := T_m \wedge R_m$ ,  $m \geq 1$ . Of course,  $X^{S_m}$  is continuous and bounded and therefore Remark 4.2 combined with (4.8) yields  $\delta^k X^{S_m} \rightarrow X$  strongly in  $B^1$  for every  $m \geq 1$ . Moreover, Lemma 4.2 says that  $\mathcal{E}_2(X^{R_m}) < \infty$  implies  $\mathcal{E}_2(X^{S_m}) < \infty$  as well. By Propositions 4.1 and 4.3 we conclude the proof.  $\square$

**Remark 4.3.** *Of course, by localizing we are able to approximate the elements in (4.9) by means of the pre-limit semimartingale sequence of Theorem 3.1. Therefore, Theorem 1.2 is an immediate consequence of Theorem 4.1.*

It follows directly from the results of this section that any continuous local strong Dirichlet process (see e.g [6]) is a local  $\delta$ -weak Dirichlet process which in turn implies that continuous semimartingales are local  $\delta$ -weak Dirichlet processes as well. Moreover, all the correspondent canonical decompositions coincides with the  $\delta$ -decompositions. As far as weak Dirichlet processes are concerned, we should stress that  $\delta$ -weak Dirichlet processes and weak Dirichlet processes are not directly comparable because of the  $\delta$ -covariation  $\langle \cdot, \cdot \rangle^\delta$  which in general does not coincides with the usual brackets in the literature [19, 13, 11].

**4.1. Some remarks about Theorem 4.1.** As far as continuous adapted processes are concerned, one should notice that the existence of  $[X, B]$  (in the sense of regularization [37]) is closely related to path regularity (e.g Hölder paths) and a restrictive probabilistic structure like strong predictability (see [13]; Corollary 3.14). Theorem 4.1 allows us to state that in the Brownian filtration case, any  $\mathbb{F}$ -adapted continuous process with finite energy along  $(\mathbb{F}^k)_{k \geq 1}$  admits the  $\delta$ -covariation w.r.t Brownian motion.

In contrast to previous works [24, 11, 13, 19] where finite energy does not guarantee a unique orthogonal decomposition, the advantage of the  $\delta$ -decomposition is that the concept of finite energy (along  $(\mathbb{F}^k)_{k \geq 1}$ ) in addition to a mild path regularity ensure a unique martingale decomposition. In fact, let  $X$  be a continuous Wiener functional such that

$$\sup_{\tau \in \mathbb{T}} \mathbb{E} \sum_{t_i \in \tau} (X_{t_i} - X_{t_{i-1}})^2 < \infty, \quad \mathcal{E}_2(X) < \infty$$

where  $\mathbb{T}$  denotes the set of dyadic partitions of  $[0, T]$ . Assume that  $[X, B]$  does not exists in the sense [11]. Then from Graversen and Rao ([24]; Th. 1) and Coquet et al. ([11], Th. 2.2) there is a natural martingale decomposition which cannot be unique. Theorem 4.1 yields the existence of a unique local  $\delta$ -decomposition.

## 5. MARTINGALE REPRESENTATION AND THE STOCHASTIC DERIVATIVE

In this section, we provide an explicit approximation scheme for the predictable martingale representation in the  $\delta$ -decomposition given in Theorem 3.1. The approximation will be given in terms of  $\mathcal{D}\delta^k$  which can be interpreted in the limit as a derivative operator on the Wiener space (see Definition 5.1) w.r.t Brownian motion. In general, we are able to prove differentiability w.r.t Brownian motion for a large class of Wiener functionals satisfying a mild integrability assumption.

For a given Wiener functional  $X$ , we introduce the following family of  $\mathbb{F}^k$ -predictable processes

$$(5.1) \quad \mathcal{D}^k X = 0 \mathbb{1}_{[[T_0^k, T_0^k]]} + \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} \delta^k X h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k]]},$$

where

$$\mathcal{D}_{T_n^k} \delta^k X h^k(s) = \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} h^k(s),$$

on  $\{T_n^k < s \leq T_{n+1}^k\}$ ,  $n \geq 1$ . Here  $h^k$  is the intensity of the jump process  $[A^k, A^k]$  given in Lemma 2.3.

We also introduce the following family of step processes

$$\mathcal{V}^k X = 0 \mathbb{1}_{[[T_0^k, T_0^k]]} + \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} \delta^k X \mathbb{1}_{[[T_n^k, T_{n+1}^k]]}.$$

One should notice that

$$\mathcal{D}_s^k X - \mathcal{V}_s^k X = \mathcal{D}_{T_n^k} \delta^k X (h^k(s) - 1) \text{ on } \{T_n^k < s \leq T_{n+1}^k\}, \quad n \geq 1$$

and therefore,  $\mathcal{D}^k X$  and  $\mathcal{V}^k X$  are identical up to the continuous function  $h^k$ . In view of Theorem 3.1, the goal of this section is to show robustness of our approximation scheme in the sense that

$$\lim_{k \rightarrow \infty} \mathcal{D}^k X = H^X \quad \text{weakly}$$

whenever  $X$  is a finite energy  $\delta$ -weak Dirichlet process with decomposition  $X = X_0 + \int_0^t H_s^X dB_s + N^X$ . More general conditions are provided for Wiener functionals with  $p$ -finite energy for  $p \neq 2$ . One should notice that since there is no a priori path regularity of  $X$  (in particular  $H^X$ ) one has to choose an appropriate topology in order to get the existence of  $\lim_{k \rightarrow \infty} \mathcal{D}^k X$ . In the sequel, we denote by  $\lambda$  the usual Lebesgue measure on  $[0, T]$  where  $0 < T < \infty$  is fixed.

One should notice that it does not make any difference to work with (5.1) or the correspondent  $\mathbb{F}^k$ -optional version

$$\mathcal{D}^{k,o} X := \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} \delta^k X h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k]]}.$$

Indeed,  $\lim_{k \rightarrow \infty} \mathcal{D}^k X$  exists weakly in  $L^1(\lambda \times \mathbb{P})$  if, and only if,  $\lim_{k \rightarrow \infty} \mathcal{D}^{k,o} X$ . In this case,  $\lim_{k \rightarrow \infty} \mathcal{D}^k X = \lim_{k \rightarrow \infty} \mathcal{D}^{k,o} X$ .

Let us begin with the following technical Lemmas.

**Lemma 5.1.** *If  $g \in L^\infty$  then for every  $1 < p < \infty$*

$$\mathbb{E} \sup_{n \geq 1} \left| \mathbb{E}[g | \mathcal{G}_n^k] - \mathbb{E}[g | \mathcal{G}_{n-1}^k] \right|^p \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* Let us recall the following pseudometric on the space of sigma-algebras

$$d(\mathcal{G}_n^k, \mathcal{G}_{n-1}^k) = \max \left\{ \sup_{E \in \mathcal{G}_n^k} \inf_{F \in \mathcal{G}_{n-1}^k} \mathbb{P}(E \Delta F), \sup_{F \in \mathcal{G}_{n-1}^k} \inf_{E \in \mathcal{G}_n^k} \mathbb{P}(E \Delta F) \right\},$$

where  $\mathcal{G}_n^k = \mathcal{G}_{n-1}^k \vee \sigma(2^{-k}\sigma_n^k; T_n^k)$ . The additional information in  $\mathcal{G}_n^k$  comes only through  $\sigma(\sigma_n^k, T_n^k)$  where  $\max\{\sup_{n \geq 1} |T_n^k - T_{n-1}^k|, \sup_{n \geq 1} |B_{T_n^k} - B_{T_{n-1}^k}|\} \rightarrow 0$  a.s as  $k \rightarrow \infty$ . Therefore, it is straightforward to check that

$$\lim_{k \rightarrow \infty} \sup_{n \geq 1} d(\mathcal{G}_n^k, \mathcal{G}_{n-1}^k) = 0.$$

By repeating the same arguments given in ([7]; Th. 3) one can easily check that in this case  $\mathbb{E}[g|\mathcal{G}_n^k] - \mathbb{E}[g|\mathcal{G}_{n-1}^k] \rightarrow 0$  as  $k \rightarrow \infty$  in probability uniformly in  $n \geq 1$ .  $\square$

In the sequel, for  $\alpha_t^k := [A^k, A^k]_t - \langle A^k, A^k \rangle_t$  we write  $\int_0^t Y_s d\alpha_s^k$  to denote the correspondent Lebesgue-Stieltjes integral w.r.t the locally integrable variation process  $\alpha^k$ .

**Lemma 5.2.** *Let  $X$  be a Wiener functional such that  $\mathcal{E}_p(X) < \infty$  for  $1 < p \leq 2$ . Then for every  $t \in [0, T]$*

$$\int_0^t \mathcal{V}_s^k X d\alpha_s^k \rightarrow 0$$

holds weakly in  $L^1$  as  $k \rightarrow \infty$ .

*Proof.* At first, one can easily check that  $\mathbb{E} \int_0^T |\mathcal{V}_s^k X| \text{Var}(\alpha^k)(ds) < \infty$  for every  $k \geq 1$ . We fix  $t \in [0, T]$ ,  $g \in L^\infty$  and let us denote by  $C$  a generic constant which may differ from line to line. From ([42]; Prop. 2) we shall write

$$\alpha_t^k = \oint_0^t \Delta A_s^k dA_s^k = \int_0^t \Delta A_s^k dA_s^k - Z_t^k,$$

where

$$\Delta A^k = \sum_{n=1}^{\infty} (B_{T_n^k} - B_{T_{n-1}^k}) \mathbb{1}_{[[T_n^k, T_n^k]]},$$

$Z_t^k := [\int \Delta A_s^k dA_s^k]_t^{p,k}$  and  $\int_0^t \Delta A_s^k dA_s^k$  is understood in the Lebesgue-Stieltjes sense.

The idea is to use  $\Delta A^k$  as a mollifier which compensates the singularity  $(B_{T_n^k} - B_{T_{n-1}^k})^{-1}$  for  $k$  sufficiently large. By the very definition

$$(5.2) \quad \int_0^t g \mathcal{V}_s^k X d\alpha_s^k = \int_0^t g \mathcal{V}_s^k X \Delta A_s^k dA_s^k$$

$$(5.3) \quad - \int_0^t g \mathcal{V}_s^k X dZ_s^k,$$

where the integrals in (5.2) and (5.3) are understood in the Lebesgue-Stieltjes sense. By taking advantage of the  $\mathbb{F}^k$ -predictability of  $\mathcal{V}^k X$  we shall use the predictable and optional dualities at the same time on  $\mathbb{F}^k$  to write (see [25] Th. 5.26)

$$\begin{aligned} \mathbb{E} \int_0^t g \mathcal{V}_s^k X d\alpha_s^k &= \mathbb{E} \int_0^t {}^{k,o}[g]_s \mathcal{V}_s^k X \Delta A_s^k dA_s^k \\ &- \mathbb{E} \int_0^t {}^{k,p}[g]_s \mathcal{V}_s^k X \Delta A_s^k dA_s^k. \end{aligned}$$



By the very definition

$$({}^{o,k}[g] - {}^{p,k}[g])\mathcal{V}^k X \Delta A^k = \sum_{n=2}^{\infty} \beta_n^k \gamma_n^k \Delta \delta^k X_{T_{n-1}^k} \mathbb{1}_{[[T_n^k, T_n^k]]},$$

where  $\beta_n^k := \mathbb{E}[g|\mathcal{G}_n^k] - \mathbb{E}[g|\mathcal{G}_{n-1}^k]$  and  $\gamma_n^k := \frac{\sigma_n^k}{\sigma_{n-1}^k}$ ;  $n \geq 2$ , where  $\sigma_n^k$  is given in (2.2). Therefore, Hölder inequality and the  $p$ -finite energy assumption yield

$$\begin{aligned} \left| \mathbb{E} \int_0^t g \mathcal{V}_s^k X d\alpha_s^k \right| &= \left| \mathbb{E} \sum_{n=2}^{\infty} \beta_n^k \gamma_n^k \Delta \delta^k X_{T_{n-1}^k} \Delta A_{T_n^k}^k \mathbb{1}_{\{T_n \leq t\}} \right| \\ &\leq \mathbb{E}^{1/p} \sum_{n=2}^{\infty} |\Delta \delta^k X_{T_{n-1}^k}|^p \mathbb{1}_{\{T_n \leq t\}} \times \mathbb{E}^{1/q} \sum_{n=2}^{\infty} |\beta_n^k \Delta A_{T_n^k}^k|^q \mathbb{1}_{\{T_n \leq t\}} \\ &\leq \mathcal{E}_p^{1/p}(X) \mathbb{E}^{1/q} \sum_{n=2}^{\infty} |\beta_n^k \Delta A_{T_n^k}^k|^q \mathbb{1}_{\{T_n \leq T\}}, \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . So now it is enough to show that

$$\mathbb{E} \sum_{n=2}^{\infty} |\beta_n^k \Delta A_{T_n^k}^k|^q \mathbb{1}_{\{T_n \leq T\}} \rightarrow 0$$

as  $k \rightarrow \infty$ . Since  $q \geq 2$ , we shall take an arbitrary  $\eta > 0$  and we use Lemma 5.1 to get

$$\begin{aligned} \mathbb{E} \sum_{n=2}^{\infty} |\beta_n^k \Delta A_{T_n^k}^k|^q \mathbb{1}_{\{T_n \leq T\}} &\leq \mathbb{E} \sup_{n \geq 1} |\beta_n^k|^q [A^k, A^k]_T \\ &\leq C\eta + 2\|g\|^q \int_{\{\sup_{n \geq 1} |\beta_n^k|^q > \eta\}} [A^k, A^k]_T d\mathbb{P} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ . In the last above inequality we have used the fact that  $[A^k, A^k]_T$  is uniformly integrable. Since  $g \in L^\infty$  was arbitrary, we may conclude the proof of the Lemma.  $\square$

Now we are in position to show the main result of this section.

**Theorem 5.1.** *If  $X$  is a Wiener functional with  $p$ -finite energy for  $1 < p \leq 2$  such that  $\langle X, B \rangle^\delta$  exists, then  $\lim_{k \rightarrow \infty} \mathcal{D}^k X$  exists weakly in  $L^1(\lambda \times \mathbb{P})$ . In particular, any finite energy  $\delta$ -weak Dirichlet process  $X$  admits a unique  $\delta$ -decomposition*

$$X_t = X_0 + \int_0^t H_s^X dB_s + N_t^X$$

where the progressive process  $H^X$  which represents the martingale part is actually given by

$$H^X = \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k]]} \quad \text{weakly in } L^1(\lambda \times \mathbb{P}).$$

*Proof.* In order to show the weak convergence in  $L^1(\lambda \times \mathbb{P})$ , it is sufficient to prove that

$$\lim_{k \rightarrow \infty} \mathbb{E} g \int_0^t \mathcal{D}_s^k X ds \quad \text{exists for every } t \in [0, T] \text{ and } g \in L^\infty.$$

At first, by the very definition

$$\begin{aligned} \mathbb{E} g \int_0^t \mathcal{D}_s^k X ds &= \mathbb{E} g \int_0^t \mathcal{V}_s^k X d\langle A^k, A^k \rangle_s \\ &= \mathbb{E} g \int_0^t \mathcal{V}_s^k X d\alpha_s^k + \mathbb{E} g \int_0^t \mathcal{V}_s^k X d[A^k, A^k]_s, \end{aligned}$$

where  $\alpha_t^k = \langle A^k, A^k \rangle_t - [A^k, A^k]_t$ . One should notice that

$$(5.4) \quad \int_0^t \mathcal{V}_s^k X d[A^k, A^k]_s = \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} \delta^k X (B_{T_{n+1}^k} - B_{T_n^k})^2 \mathbb{1}_{\{T_{n+1}^k \leq t\}}.$$

On the other hand, by assumption

$$(5.5) \quad \lim_{k \rightarrow \infty} [\delta^k X, A^k]_t = \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} (\mathcal{D}_{T_n^k} \delta^k X) (B_{T_n^k} - B_{T_{n-1}^k})^2 \mathbb{1}_{\{T_n^k \leq t\}} = \langle X, B \rangle_t^\delta \quad \text{weakly in } L^1(\mathbb{P}).$$

By construction  $(B_{T_{n+1}^k} - B_{T_n^k})^2 = (B_{T_i^k} - B_{T_{i-1}^k})^2$  for every  $n \geq 0$  and  $i \geq 1$ . By rewriting (5.4) and using (5.5) it follows that

$$\begin{aligned} \int_0^t \mathcal{V}_s^k X d[A^k, A^k]_s &= [\delta^k X, A^k]_t \\ &\quad - \sum_{i=1}^{\infty} \mathcal{D}_{T_i^k} \delta^k X (B_{T_i^k} - B_{T_{i-1}^k})^2 \mathbb{1}_{\{T_{i-1}^k \leq t < T_i^k\}} \rightarrow \langle X, B \rangle_t^\delta \end{aligned}$$

weakly in  $L^1(\mathbb{P})$ . This convergence together with Lemma 5.2 allow us to conclude that

$$(5.6) \quad \lim_{k \rightarrow \infty} \mathbb{E} g \int_0^t \mathcal{D}_s^k X ds = \mathbb{E} g \langle X, B \rangle_t^\delta.$$

Since  $g \in L^\infty$  and  $t \in [0, T]$  were arbitrary, we may conclude that  $\lim_{k \rightarrow \infty} \mathcal{D}^k X$  exists weakly in  $L^1(\lambda \times \mathbb{P})$ . In the particular finite energy case,  $X$  is a  $\delta$ -weak Dirichlet process  $X = X_0 + M^X + N^X$  where  $M^X \in \mathcal{H}^2$  and  $\langle N^X, B \rangle_t^\delta = 0$ . In this case, the linearity of  $\mathcal{D}^k$  yields

$$\mathcal{D}^k X = \mathcal{D}^k M^X + \mathcal{D}^k N^X.$$

By the orthogonality of  $N^X$  w.r.t Brownian motion and using (5.6) it follows that

$$\lim_{k \rightarrow \infty} \mathcal{D}^k N^X = 0 \quad \text{weakly in } L^1(\lambda \times \mathbb{P}).$$

Since  $M^X$  is a square integrable martingale there exists a progressive process  $H^X$  such that  $M_t^X = \int_0^t H_s^X dB_s$ . In this case, Corollary 3.2 yields  $\langle M^X, B \rangle_t^\delta = \int_0^t H_s^X ds$  and therefore we shall use again relation (5.6) to conclude that

$$\lim_{k \rightarrow \infty} \mathbb{E} g \int_0^t \mathcal{D}_s^k M^X ds = \mathbb{E} g \int_0^t H_s^X ds,$$

for every  $g \in L^\infty$  and  $t \in [0, T]$ . This allows us to conclude that  $\lim_{k \rightarrow \infty} \mathcal{D}^k M^X = H^X$  and therefore  $\lim_{k \rightarrow \infty} \mathcal{D}^k X = H^X$  weakly in  $L^1(\lambda \times \mathbb{P})$ .  $\square$

**Remark 5.1.** *We notice that any Wiener functional satisfying the assumptions of Theorem 3.1 is a finite energy  $\delta$ -weak Dirichlet process. Therefore, Theorem 1.1 is an immediate consequence of Theorem 5.1.*

In view of the above result, it is natural to introduce the following notion of stochastic derivative.

**Definition 5.1.** *We say that a Wiener functional  $X$  is **differentiable w.r.t the Brownian motion** if the following limit*

$$\lim_{k \rightarrow \infty} \mathcal{D}^k X$$

*exists weakly in  $L^1(\lambda \times \mathbb{P})$ . If this is the case, we define*

$$\mathcal{D}X := \lim_{k \rightarrow \infty} \mathcal{D}^k X = \lim_{k \rightarrow \infty} \mathcal{D}^{k,o} X.$$

Therefore, any process which satisfies the mild conditions in Theorem 5.1 is differentiable w.r.t Brownian motion. Although the existence of  $\mathcal{D}X$  has its own importance when  $p \neq 2$  in Theorem 5.1, the most interesting case happens when  $X$  has finite energy. By applying Theorem 5.1 to the classical Itô representation theorem we arrive at the following result.

**Theorem 5.2.** *If  $F$  is an  $\mathcal{F}_T$ -square integrable random variable then*

$$F = \mathbb{E}[F] + \int_0^T \mathcal{D}_s F dB_s,$$

*where*

$$(5.7) \quad \mathcal{D}F = \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\mathbb{E}[F|\mathcal{G}_n^k] - \mathbb{E}[F|\mathcal{G}_{n-1}^k]}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[}$$

*weakly in  $L^1(\lambda \times \mathbb{P})$ . Therefore, Clark-Ocone formula implies that  $\mathcal{D}_t F = \mathbb{E}[D_t F | \mathcal{F}_t]$  where  $D$  denotes the Gross-Sobolev derivative of  $F$  in  $L^2(\mathbb{P})$ .*

**Remark 5.2.** *If  $F$  is not differentiable in the sense of Malliavin calculus, the Gross-Sobolev derivative  $D_t F$  is interpreted as a generalized process where  $\mathbb{E}[D_t F | \mathcal{F}_t]$  can be interpreted as a real process in  $L^2(\lambda \times \mathbb{P})$ . See [2, 5] for more details.*

## 6. EXAMPLES OF REPRESENTATIONS AND DERIVATIVES

In this section, we apply the abstract framework developed in the last sections to concrete Wiener functionals.

### 6.1. First passage time martingale representation.

Let  $\{|B_t|; \mathbb{F}; t \geq 0\}$  be the reflected Brownian motion, where by Tanaka formula the following representation holds

$$|B_t| = \int_0^t \text{sgn}(B_u) dB_u + 2L_t^0,$$

where  $L_t^0$  is the local time at zero. Let us fix an arbitrary  $\alpha > 0$  and consider

$$T_\alpha := \inf\{t > 0; |B_t| = \alpha\}.$$

It is well-known that  $T_\alpha$  has the same distribution as  $\alpha^2 T_1$  where  $T_1 = \inf\{t > 0; |B_t| = 1\}$ . Moreover, the second moment is given by  $\mathbb{E}|T_\alpha|^2 = c\alpha^4 + \alpha^2$  for some constant  $c > 0$ . Let  $\Lambda_t(\alpha) = \mathbb{E}[T_\alpha | \mathcal{F}_t]$ ,  $0 \leq t < \infty$  be the square integrable Lévy martingale generated by  $T_\alpha$ . By Theorem 5.2 we know that if  $0 < T < \infty$  then

$$(6.1) \quad \Lambda_T(\alpha) = \alpha^2 + \int_0^T \mathcal{D}_s \Lambda_T(\alpha) dB_s.$$

The abstract representation (5.7) for  $\mathcal{D}\Lambda_T(\alpha)$  can be written as follows. At first, we notice that

$$\mathbb{E}[T_\alpha | \mathcal{G}_n^k] = T_n^k + \inf\{t > T_n^k; |B_t| = \alpha\} \text{ on } \{T_n^k < T_\alpha\}$$

and therefore

$$\mathbb{E}[T_\alpha | \mathcal{G}_n^k] - \mathbb{E}[T_\alpha | \mathcal{G}_{n-1}^k] = T_n^k - T_{n-1}^k \text{ on } \{T_{n-1}^k < T_n^k < T_\alpha\}.$$

Now we pick up a set  $\Omega^*$  of full probability and consider a sequence of random variables on  $\Omega^*$  given by  $N_\alpha^k(\omega) := m$  if and only if  $T_m^k(\omega) < T_\alpha(\omega) \leq T_{m+1}^k(\omega)$ . Therefore, we shall write

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}[T_\alpha | \mathcal{G}_n^k] - \mathbb{E}[T_\alpha | \mathcal{G}_{n-1}^k]}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[} = \sum_{n=1}^{N_\alpha^k} \frac{T_n^k - T_{n-1}^k}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[},$$

and we notice that  $\mathcal{D}\Lambda_T(\alpha)$  does not really depend on  $0 < T < \infty$ . By taking  $T \rightarrow \infty$  in (6.1) and using Theorem 5.2 it follows that

$$\sum_{n=1}^{N_\alpha^k} \frac{T_n^k - T_{n-1}^k}{B_{T_n^k} - B_{T_{n-1}^k}} h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k[[} \rightarrow \mathcal{D}T_\alpha$$

weakly in  $L^1(\lambda \times \mathbb{P})$  on  $\mathbb{R} \times \Omega$ , where

$$T_\alpha = \alpha^2 + \int_0^\infty \mathcal{D}_s T_\alpha dB_s.$$

**6.2. Change of variables formula for  $\delta$ -weak Dirichlet processes.** In this section, we provide a change of variables formula for a class of finite energy  $\delta$ -weak Dirichlet processes. We fix a continuous Wiener functional  $X$  which satisfies condition (BUI) and  $\mathcal{E}_2^s(X) < \infty$ . Next, our goal is to show a chain rule for the stochastic derivative operator  $\mathcal{D}$ . In order to avoid the somewhat complicated conditional expectation structure which arises from the definition of  $\mathcal{D}$ , we will work with the following sequence of  $\mathbb{F}^k$ -predictable processes

$$(6.2) \quad \mathbb{D}^k G(X) := 0 \mathbb{1}_{[[T_0^k, T_0^k]]} + \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} G(\delta^k X) h^k \mathbb{1}_{[[T_n^k, T_{n+1}^k]]},$$

where

$$\mathcal{D}G(\delta^k X) := \sum_{n=1}^{\infty} \frac{G(\delta^k X_{T_n^k}) - G(\delta^k X_{T_{n-1}^k})}{B_{T_n^k} - B_{T_{n-1}^k}} \mathbb{1}_{[[T_n^k, T_n^k]]}, \quad k \geq 1.$$

The next Lemma states that in the particular case of a Wiener functional of the form  $G(X)$ , we shall also work with (6.2) in order to compute  $\mathcal{D}G(X)$ . The proof is almost identical to Lemma 5.2 and Theorem 5.1. For the sake of completeness we give the details here.

**Lemma 6.1.** *If  $G$  is Lipschitz then the following convergences hold true*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^t \mathbb{D}_s^k G(X) ds &= \lim_{k \rightarrow \infty} [G(\delta^k X), A^k]_t \\ &= \langle G(X), B \rangle_t^\delta \\ &= \lim_{k \rightarrow \infty} \int_0^t \mathcal{D}_s^k G(X) ds, \quad \text{weakly in } L^1 \quad 0 \leq t \leq T. \end{aligned}$$

*Proof.* Since  $G$  is Lipschitz and  $\mathcal{E}_2^s(X) < \infty$  we have

$$\begin{aligned} \sup_{k \geq 1} \mathbb{E}[G(\delta^k X), G(\delta^k X)]_T &= \sup_{k \geq 1} \mathbb{E} \sum_{n=1}^{\infty} |\Delta G(\delta^k X_{T_n^k})|^2 \mathbb{1}_{\{T_n^k \leq T\}} \\ (6.3) \quad &\leq \|G\|_{Lip} \mathcal{E}_2(X) < \infty. \end{aligned}$$

In this case, we shall apply the same arguments used in Lemma 5.2 to prove that for each  $0 \leq t \leq T$

$$\int_0^t \mathbb{V}_s^k G(X) d\alpha_s^k \rightarrow 0 \quad \text{weakly in } L^1,$$

where  $\mathbb{V}^k G(X) = 0 \mathbb{1}_{[[T_0^k, T_0^k]]} + \sum_{n=0}^{\infty} \mathcal{D}_{T_n^k} G(\delta^k X) \mathbb{1}_{[[T_n^k, T_{n+1}^k]]}$ ,  $k \geq 1$  and  $\alpha^k$  as in Lemma 5.2. Moreover, the same argument used in Theorem 5.1 also applies to  $\mathbb{V}^k G(X)$  in order to prove that for each  $0 \leq t \leq T$ ,

$$(6.4) \quad \lim_{k \rightarrow \infty} \int_0^t \mathbb{V}_s^k G(X) d[A^k, A^k]_s = \lim_{k \rightarrow \infty} [G(\delta^k X), A^k]_t \quad \text{weakly in } L^1,$$

provided that the second limit in (6.4) exists. In fact, since  $G$  is Lipschitz and  $X$  satisfies condition (BUI) we do have  $G(\delta^k X) \rightarrow G(X)$  strongly in  $B^1$ . Moreover, since  $G(\delta^k X)$  is a step process such that  $\{\Delta G(\delta^k X) \neq 0\} \subset \cup_{n=1}^{\infty} [[T_n^k, T_n^k]]$  and it satisfies (6.3), we may apply the same arguments used in Proposition 4.1 to show that

$$\lim_{k \rightarrow \infty} [G(\delta^k X), A^k]_t \text{ exists weakly in } L^1,$$

for each  $0 \leq t \leq T$ . Now since  $G(\delta^k X) - \delta^k G(X) \rightarrow 0$  strongly in  $B^1$  we do have

$$\lim_{k \rightarrow \infty} [G(\delta^k X) - \delta^k G(X), A^k]_t = 0 \text{ weakly in } L^1$$

for each  $0 \leq t \leq T$ . Therefore,

$$\begin{aligned} \lim_{k \rightarrow \infty} [G(\delta^k X), A^k]_t &= \lim_{k \rightarrow \infty} [\delta^k G(X), A^k]_t \\ &= \langle G(X), B \rangle_t^\delta \text{ weakly in } L^1. \end{aligned}$$

Finally, by writing

$$\begin{aligned} \int_0^t \mathbb{D}_s^k G(X) ds &= \int_0^t \mathbb{V}_s^k G(X) d\langle A^k, A^k \rangle_s \\ &= \int_0^t \mathbb{V}_s^k G(X) d\alpha_s^k + \int_0^t \mathbb{V}_s^k G(X) d[A^k, A^k]_s, \end{aligned}$$

and summing up the above conclusions we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^t \mathbb{D}_s^k G(X) ds &= \lim_{k \rightarrow \infty} [G(\delta^k X), A^k]_t \\ &= \lim_{k \rightarrow \infty} [\delta^k G(X), A^k]_t \\ &= \langle G(X), B \rangle_t^\delta \text{ weakly in } L^1, \quad 0 \leq t \leq T. \end{aligned}$$

□

By the very definition of the weak topology of  $L^1(\lambda \times \mathbb{P})$ , one should notice that

$$\lim_{k \rightarrow \infty} \mathbb{D}^k G(X) = \lim_{k \rightarrow \infty} \mathbb{D}^{o,k} G(X)$$

provided one of the limits exists. Here  $\mathbb{D}^{o,k} G(X)$  is the optional version of  $\mathbb{D}^k G(X)$ . Now we are able to prove the main result of this section. We study the decomposition for  $C^1$ -real-valued functions with bounded derivatives.

**Theorem 6.1.** *Let  $X = X_0 + M + Z$  be a continuous  $\delta$ -weak Dirichlet process satisfying (BUI) and such that  $\mathcal{E}_2^s(X) < \infty$ . Assume that  $F$  is a  $C^1$ -real valued function with bounded derivative  $f$ . Then the process  $F(X)$  is a finite-energy  $\delta$ -weak Dirichlet process and the following decomposition holds*

$$(6.5) \quad F(X_t) = X_0 + \int_0^t f(X_s) dM_s + V_t, \quad 0 \leq t \leq T.$$

The non-martingale part is given by

$$(6.6) \quad V = \lim_{k \rightarrow \infty} \int_0^\cdot U_s^{k,F} h_s^k ds \quad \text{weakly in } B^1$$

where

$$(6.7) \quad U_t^{k,F} = 0 \mathbb{1}_{\{T_0^k=t\}} + \frac{1}{2^{-2k}} \sum_{n=1}^{\infty} \mathbb{E}[F(X_t) - F(X_{T_{n-1}^k}) | \mathcal{G}_{n-1}^k; T_n^k = t] \mathbb{1}_{\{T_{n-1}^k < t \leq T_n^k\}}$$

*Proof.* In order to prove our statement, we fix  $0 \leq t \leq T$  and  $g \in L^\infty$ . Throughout this proof  $C$  will denote a generic constant which may differ from line to line. From Proposition 4.1 and Theorem 5.1 we already know that

$$(6.8) \quad F(X_t) = F(X_0) + \int_0^t \mathcal{D}_s F(X) dB_s + V_t,$$

where by Theorem 3.1 the non-martingale part  $V$  is given by the limit (6.6) and (6.7). So in view of Lemma 6.1 and the definition of the weak topology of  $L^1(\lambda \times \mathbb{P})$ , in order to show (6.5) it is sufficient to prove that

$$(6.9) \quad \lim_{k \rightarrow \infty} \mathbb{E} g \int_0^t \mathbb{D}_s^{o,k} F(X) ds = \mathbb{E} g \int_0^t f(X_s) \mathcal{D}_s X ds,$$

where  $\mathbb{D}^{o,k} F(X)$  is the optional version of  $\mathbb{D}^k F(X)$  given in (6.2). In order to show (6.9), we fix  $\varepsilon > 0$  and we write the increments by means of a Taylor expansion as follows

$$F(\delta^k X_{T_n^k}) - F(\delta^k X_{T_{n-1}^k}) = f(\delta^k X_{T_{n-1}^k}) \Delta \delta^k X_{T_n^k} + R^\varepsilon(\delta^k X_{T_{n-1}^k}, \delta^k X_{T_n^k}) \quad \text{on } \{T_n^k \leq t\}$$

where  $R^\varepsilon(\delta^k X_{T_{n-1}^k}, \delta^k X_{T_n^k}) := C_{k,n}^\varepsilon \Delta \delta^k X_{T_n^k}$  and  $C_{k,n}^\varepsilon$  satisfies

$$(6.10) \quad |C_{k,n}^\varepsilon| \leq \sup_{x \in \mathbb{R}} |f(x)|$$

$$(6.11) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{k \rightarrow \infty} \mathbb{P} \left\{ \sup_{n \geq 1} |C_{k,n}^\varepsilon| \mathbb{1}_{\{T_n^k \leq t\}} > \eta \right\} = 0 \quad \forall \eta > 0.$$

From the Taylor formula we have

$$\begin{aligned}
\mathbb{D}_s^{o,k} F(X) &= \sum_{n=1}^{\infty} \frac{f(\delta^k X_{T_{n-1}^k}) \Delta \delta^k X_{T_n^k}}{\Delta A_{T_n^k}^k} h_s^k \mathbb{1}_{\{T_n^k \leq s < T_{n+1}^k\}} \\
&+ \sum_{n=1}^{\infty} \frac{R^\varepsilon(\delta^k X_{T_{n-1}^k}, \delta^k X_{T_n^k})}{\Delta A_{T_n^k}^k} h_s^k \mathbb{1}_{\{T_n^k \leq s < T_{n+1}^k\}} \\
&=: J_1^k(s) + J_2^k(s) \quad 0 \leq s \leq T,
\end{aligned}$$

so we write

$$(6.12) \quad \mathbb{E}g \int_0^t \mathbb{D}_s^{o,k} F(X) ds = \mathbb{E}g \int_0^t J_1^k(s) ds + \mathbb{E}g \int_0^t J_2^k(s) ds.$$

The second term can be estimated as follows

$$\begin{aligned}
\left| \mathbb{E}g \int_0^t J_2^k(s) ds \right| &\leq \|f\| \|g\| \int_{\{\sup_{k,n} |C_{k,n}^\varepsilon| > \eta\} \times [0, T]} |\mathcal{D}^{o,k} X| d(\lambda \times \mathbb{P}) \\
&+ \|g\| \eta \int_{\Omega \times [0, T]} |\mathcal{D}^{o,k} X| d(\lambda \times \mathbb{P})
\end{aligned}$$

for a given  $\eta > 0$ . From (6.10), (6.11) and the uniformly integrability of  $\mathcal{D}^{o,k} X$  we may take  $\varepsilon \downarrow 0$  and  $k \rightarrow \infty$  to conclude that the second term in (6.12) vanishes.

Now let us estimate the first term. We notice that we shall write

$$\int_0^t J_1^k(s) ds = \int_0^t f(X_s^{k-}) \mathcal{D}_s^{o,k} X ds,$$

where

$$X^{k-} = \sum_{n=1}^{\infty} \delta^k X_{T_{n-1}^k} \mathbb{1}_{[[T_n^k, T_{n+1}^k[[[ \cdot}$$

Let

$$\begin{aligned}
(6.13) \quad \mathbb{E}g \int_0^t J_1^k(s) ds &= \mathbb{E}g \int_0^t \left\{ f(X_s^{k-}) - f(X_s) \right\} \mathcal{D}_s^{o,k} X ds \\
&+ \mathbb{E}g \int_0^t f(X_s) \mathcal{D}_s^{o,k} X ds.
\end{aligned}$$

Since we know that  $\mathcal{D}^{o,k} \rightarrow \mathcal{D}X$  weakly in  $L^1(\lambda \times \mathbb{P})$  and  $gf(X_\cdot) \in L^\infty(\lambda \times \mathbb{P})$ , we do have

$$\mathbb{E} \int_0^t gf(X_s) \mathcal{D}_s^{o,k} X ds \rightarrow \mathbb{E} \int_0^t gf(X_s) \mathcal{D}_s X ds \quad \text{as } k \rightarrow \infty.$$

The first term in (6.13) can be estimated as follows



$$\begin{aligned}
\left| \mathbb{E} g \int_0^t \left\{ f(X_s^{k-}) - f(X_s) \right\} \mathcal{D}_s^{o,k} X ds \right| &\leq C \int_{\Lambda^k(\eta) \times [0, T]} |\mathcal{D}^{o,k} X| d(\lambda \times \mathbb{P}) \\
&+ \eta \|g\| \int_{\Omega \times [0, T]} |\mathcal{D}^{o,k} X| d(\lambda \times \mathbb{P}).
\end{aligned}$$

where  $\Lambda^k(\eta) = \{\sup_{0 \leq s \leq T} |f(X_s^{k-}) - f(X_s)| > \eta\}$  for a given  $\eta > 0$ . Since  $f(X_s^{k-}) \rightarrow f(X_s)$  uniformly in probability and  $\mathcal{D}^{o,k} X$  is uniformly integrable on  $L^1(\lambda \times \mathbb{P})$  we do have

$$\mathbb{E} g \int_0^t J_1^k(s) ds \rightarrow \mathbb{E} \int_0^t g f(X_s) \mathcal{D}_s X ds.$$

This shows (6.9) and therefore Lemma 6.1 yields

$$(6.14) \quad \lim_{k \rightarrow \infty} \mathbb{D}^{o,k} F(X) = \mathcal{D}F(X) = f(X) \mathcal{D}X.$$

Identities (6.14) and (6.8) allow us to conclude the proof.  $\square$

**Remark 6.1.** Under the same assumption of Theorem 6.1 but  $F : \mathbb{R} \rightarrow \mathbb{R}$  assumed only Lipschitz, we have  $F(X)$  is a finite energy  $\delta$ -weak Dirichlet process as well.

### 6.3. Fractional functionals.

In this section we illustrate that our smooth semimartingale approximation scheme applies to a significant particular case. We suppose that the Wiener functional is induced by the fractional Brownian motion  $B^H$  (henceforth abbreviated by fBm). In this case, as discussed in previous sections, the limit will not be a semimartingale but the pre-limit sequence are given by  $\mathbb{F}^k$ -special semimartingales and hence with a much simpler probabilistic structure. Similar types of approximations have been recently proposed in the Mathematical Finance context. See the works [28], [40], [39] and other references therein. Contrary to previous works, we are able to explicitly identify the semimartingale structure of the discrete skeleton in its own natural filtration  $\mathbb{F}^k$  which converges to the filtration generated by fBm. Moreover, we start with the fractional functional itself instead of an external process with given parameters (e.g. Poisson and renewal processes). Such result is a straightforward consequence of Theorem 3.1 and Theorem 6.1.

In view of the finite energy assumption in Theorem 3.1, we restrict our analysis to the case  $H > 1/2$ . The fBm admits the following Volterra representation

$$B_t^H = \int_0^t K(t, s) dB_s,$$

where  $K(t, s) = cs^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$  for  $0 < s \leq t \leq T$  and  $c$  is a positive constant. By the very definition,  $B$  and  $B^H$  generates the same filtration  $\mathbb{F}$ .

**Proposition 6.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function of class  $C^1(\mathbb{R})$  with bounded derivative. Let  $M^{k,H}$  and  $N^{k,H}$  be the correspondent canonical  $\mathbb{F}^k$ -decomposition of  $\delta^k f(B^H)$ . Then  $f(B^H)$  is a finite energy  $\delta$ -weak Dirichlet process where

$$M^{k,H} = \oint_0^\cdot \mathcal{D}_s \delta^k f(B^H) dA_s^k \rightarrow 0,$$

$$N^{k,H} = \frac{1}{2^{-2k}} \sum_{n=1}^{\infty} \int_0^\cdot \mathbb{E}[f(B_s^H) - f(B_{T_{n-1}^k}^H) | \mathcal{G}_{n-1}^k; T_n^k = s] h_s^k \mathbb{1}_{\{T_n^k < s \leq T_{n+1}^k\}} ds \rightarrow f(B^H)$$

weakly in  $B^1$  as  $k \rightarrow \infty$ .

*Proof.* Since  $B^H \in B^2$  has continuous paths then it satisfies condition (BUI) and therefore  $\delta^k f(B^H) \rightarrow f(B^H)$  strongly in  $B^1$ . In view of Theorem 6.1 and Proposition 4.1, we only have to check that  $\mathcal{E}_2(B^H) < \infty$ . But this is a simple application of Fernique's Theorem together with the fact that fBm admits  $\gamma$ -Hölder continuous paths for  $\gamma > 1/2$ . In this case, similar to the deterministic partition case we have

$$\mathbb{E} \left[ \sum_{n=1}^{\infty} \left( B_{T_n^k}^H - B_{T_{n-1}^k}^H \right)^2 \mathbb{1}_{\{T_n^k \leq T\}} \right] \rightarrow 0.$$

By applying Theorem 3.1 we conclude the proof.  $\square$

**Remark 6.2.** A natural issue regarding the above approximation is about completeness and no-arbitrage structure for the skeleton  $(\delta^k X, \mathbb{F}^k)$  when  $X$  is a geometric fBm. Related questions have been discussed in [28, 40, 39] starting with a given renewal counting process with suitable parameters. We believe that the financial market arising from the embedded semimartingale structure  $(\delta^k X, \mathbb{F}^k)$  can potentially be useful to understand the fine structure of a fractional market due to the very simple sample paths of  $\delta^k X$  where the martingale part only jumps at  $(T_n^k)_{n \geq 1}$ .

## 7. DISCRETE APPROXIMATIONS OF THE ITÔ FORMULA AND INTEGRALS W.R.T LOCAL TIMES

In the remainder of this section, we fix  $y \in \mathbb{R}$  and we consider the Brownian motion  $B$  starting from  $B_0 = y$ . We begin by identifying the canonical representation of the  $\mathbb{F}^k$ -special semimartingale  $\delta^k F(B)$ . It is a simple exercise the application of Proposition 2.1 to get the following result.

**Lemma 7.1.** Assume that  $F$  is a locally bounded function. Then the  $\mathbb{F}^k$ -decomposition  $(M^{k,F}, N^{k,F})$  of  $F(B)$  is given by

$$F(A_t^k) = F(A_0^k) + \oint_0^t \mathcal{D}_s F(A^k) dA_s^k + \int_0^t U_s^{k,F} h_s^k ds,$$

where

$$(7.1) \quad U_t^{k,F} = \frac{1}{2} \sum_{i=-1}^1 \frac{\{F(A_{t-}^k + i2^{-k}) - F(A_{t-}^k)\}}{2^{-2k}}, \quad 0 \leq t \leq T.$$

Let us now study under what conditions one may apply Theorem 3.1 developed in the last section to the Wiener function  $F(B)$ . In the sequel, we make use of the following sequence of  $\mathbb{F}$ -stopping times given by

$$S_m = \inf\{t > 0; |B_t| = 2^m\}; \quad m \geq 1.$$

One should notice that  $S_m$  is an  $\mathbb{F}^k$ -stopping time for every  $k > m$ . Our method is based on the construction of the Brownian local time via the random walk  $A^k$ , see Knight [30, 29] and other references therein. Let  $i2^{-k}$  be a partition of  $[-2^m, 2^m]$  for  $i = -2^{m+k}, \dots, -1, 0, 1, 2^{m+k}$ . With this partition at hand, we define the following process on  $[0, T] \times [-2^m, 2^m]$

$$\eta^{k,m}(t, \cdot) := \sum_{j=-2^{m+k}+1}^{2^{m+k}} \left( \eta_{\{j-1,j\}}^{k,m}(t, \cdot) + \eta_{\{j,j-1\}}^{k,m}(t, \cdot) \right) \mathbb{1}_{[(j-1)2^{-k}, j2^{-k})},$$

where

$$\eta_{\{j,j-1\}}^{k,m}(t, x) := \sum_{n=1}^{\infty} \mathbb{1}_{\{A_{T_{n-1}^k}^k = j2^{-k}, A_{T_n^k}^k = (j-1)2^{-k}\}} \mathbb{1}_{\{T_n^k \leq t \wedge S_m\}},$$

$$\eta_{\{j-1,j\}}^{k,m}(t, x) := \sum_{n=1}^{\infty} \mathbb{1}_{\{A_{T_{n-1}^k}^k = (j-1)2^{-k}, A_{T_n^k}^k = j2^{-k}\}} \mathbb{1}_{\{T_n^k \leq t \wedge S_m\}},$$

for  $(j-1)2^{-k} \leq x < j2^{-k}$  and  $0 \leq t \leq T$ . To shorten notation, we write  $\Delta_k^j := F(j2^{-k}) - F((j-1)2^{-k})$  for  $j = -2^{m+k} + 1, \dots, -1, 0, 1, 2^{m+k}$ . By the very definition, we notice that

$$\begin{aligned} [F(A^k), A^k]_{t \wedge S_m} &= \sum_{n=1}^{\infty} \Delta F(A_{T_n^k}^k) \Delta A_{T_n^k}^k \mathbb{1}_{\{T_n^k \leq t \wedge S_m\}} \\ &= \sum_{j=-2^{m+k}+1}^{2^{m+k}} 2^{-k} \sum_{n=1}^{\infty} \mathbb{1}_{\{A_{T_{n-1}^k}^k = (j-1)2^{-k}, A_{T_n^k}^k = j2^{-k}\}} \mathbb{1}_{\{T_n^k \leq t \wedge S_m\}} \Delta_k^j \\ &\quad + \sum_{j=-2^{m+k}+1}^{2^{m+k}} 2^{-k} \sum_{n=1}^{\infty} \mathbb{1}_{\{A_{T_{n-1}^k}^k = j2^{-k}, A_{T_n^k}^k = (j-1)2^{-k}\}} \mathbb{1}_{\{T_n^k \leq t \wedge S_m\}} \Delta_k^j. \end{aligned}$$

The same arguments applies to  $[F(A^k), F(A^k)]$  and therefore we arrive at the following result.

**Lemma 7.2.** *If  $F$  is a real Borel function then*

$$(7.2) \quad [F(A^k), A^k]_{t \wedge S_m} = \sum_{j=-2^{m+k}+1}^{2^{m+k}} 2^{-k} \eta^{k,m}(t, (j-1)2^{-k}) \Delta_k^j,$$

$$(7.3) \quad [F(A^k), F(A^k)]_{t \wedge S_m} = \sum_{j=-2^{m+k}+1}^{2^{m+k}} \left[ \frac{\Delta_k^j}{2^{-k}} \right]^2 2^{-2k} \eta^{k,m}(t, (j-1)2^{-k}),$$

for every  $t \in [0, T]$ .

In the sequel,  $L_t^x$  denotes the Brownian local time process. A straightforward application of ([29]; Th 0.2) and ([3]; Prop. 2.1) gives the following result.

**Lemma 7.3.** *For every  $t \in [0, T]$  and  $m \geq 1$ ,*

$$\lim_{k \rightarrow \infty} 2^{-k} \eta^{k,m}(t, x) = 2L_{t \wedge S_m}^x$$

*in  $L^1(\mathbb{P})$  and a.s uniformly in  $x$ .*

We are now in position to prove the following result. In the sequel, if  $F$  is a bounded variation function then we denote  $\mu_F$  the correspondent Lebesgue-Stieljtes measure.

**Lemma 7.4.** *If  $F$  has bounded variation then the covariation process  $\langle F(B), B^{S_m} \rangle^\delta$  exists. In this case,*

$$(7.4) \quad \langle F(B), B^{S_m} \rangle_t^\delta = \int_{-\infty}^{+\infty} 2L_{t \wedge S_m}^x \mu_F(dx).$$

*In particular, if  $F(x) = \int_0^x f(y)dy$  is absolutely continuous then*

$$(7.5) \quad \lim_{m \rightarrow \infty} \langle F(B), B \rangle_{t \wedge S_m}^\delta = \int_0^t f(B_s)ds \quad a.s., \quad 0 \leq t \leq T.$$

*Proof.* Assume that  $F$  has finite variation and let us fix  $t \in [0, T]$  and  $m \geq 1$ . From (7.2) we shall write

$$\begin{aligned} [F(A^k), A^k]_{t \wedge S_m} &= \sum_{j=-2^{m+k}+1}^{2^{m+k}} \left[ 2^{-k} \eta^{k,m}(t, (j-1)2^{-k}) - 2L_{t \wedge S_m}^{(j-1)2^{-k}} \right] \Delta_k^j \\ &+ \sum_{j=-2^{m+k}+1}^{2^{m+k}} 2L_{t \wedge S_m}^{(j-1)2^{-k}} \Delta_k^j \\ &= T_1^{k,m}(t) + T_2^{k,m}(t). \end{aligned}$$

By Lemma 7.3 and the bounded variation assumption it follows that  $\lim_{k \rightarrow \infty} T_1^{k,m}(t) = 0$  a.s and in  $L^1(\mathbb{P})$ . Moreover, the continuity of  $L_{t \wedge S_m}$  yields

$$\lim_{k \rightarrow \infty} T_2^{k,m}(t) = 2 \int_{-2^m}^{2^m} L_{t \wedge S_m}^x \mu_F(dx), \quad a.s \text{ and in } L^1(\mathbb{P}).$$

Since  $L_{t \wedge S_m}^x$  is continuous and zero outside the interval  $[-2^m, 2^m]$ , we have that

$$\int_{-2^m}^{2^m} L_{t \wedge S_m}^x \mu_F(dx) = \int_{-\infty}^{\infty} L_{t \wedge S_m}^x \mu_F(dx).$$

From the above convergence, we conclude that the identity (7.4) holds. In particular, if  $F$  is absolutely continuous then Engelbert-Schmidt Zero-One law yields

$$\lim_{m \rightarrow \infty} \langle F(B), B \rangle_{t \wedge S_m}^\delta = \lim_{m \rightarrow \infty} \int_0^{t \wedge S_m} f(B_s)ds = \int_0^t f(B_s)ds, \quad a.s \quad 0 \leq t \leq T.$$

□

**Lemma 7.5.** *Assume that  $F$  is an absolutely continuous function with locally square-integrable derivative. Then the process  $F(B)$  has 2-finite energy (locally) and*

$$(7.6) \quad \langle F(B), F(B) \rangle_t^\delta = \int_0^t |f(B_s)|^2 ds, \quad \text{if } f \in L^2(\mathbb{R}).$$

*Proof.* At first, we fix  $m \geq 1$  and we prove that  $f \in L^2([-2^m, 2^m])$  implies that the process  $\{F(B_{t \wedge S_m}); 0 \leq t \leq T\}$  has finite energy in the sense of Definition 3.1. In the sequel, we denote by  $C$  a positive constant which may defer from line to line. If  $g \in L^2([-2^m, 2^m])$ , we shall define

$$Mg(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |g(y)| dy \quad -2^m \leq x \leq 2^m.$$

It is well known that  $\|Mg\|_{L^2} \leq C \|g\|_{L^2}$  for some constant  $C$ . Then, it follows from (7.3) that

$$\begin{aligned} \mathbb{E}[F(A^k), F(A^k)]_{t \wedge S_m} &\leq \sum_{j=-2^{m+k}+1}^{2^{m+k}} |Mf((j-1)2^{-k})|^2 \mathbb{E}[2^{-k} \eta^{k,m}(t \wedge S_m, (j-1)2^{-k})] 2^{-k} \\ &\leq C \sum_{j=-2^{m+k}+1}^{2^{m+k}} [Mf((j-1)2^{-k})]^2 2^{-k} \\ &\leq C \|Mf\|_{L^2}^2 \leq C \|f\|_{L^2}^2. \end{aligned}$$

This shows that  $\mathcal{E}_2(F(B)) < \infty$  (locally) if  $f \in L^2$  ( $f \in L_{loc}^2$ ). It remains to establish (7.6). For this, let  $m \geq 1$  and  $t \in [0, T]$  and we consider the following approximating sequence in  $L^2[-2^m, 2^m]$

$$g^k := \sum_{j=-2^{m+k}+1}^{2^{m+k}} \frac{\Delta_k^j}{2^{-k}} \mathbb{1}_{[(j-1)2^{-k}, j2^{-k})}.$$

The Lebesgue differentiation theorem yields  $\lim_{k \rightarrow \infty} g^k = f$  a.s and a simple application of Fatou lemma gives

$$(7.7) \quad \sup_{k \geq 1} \int_{-2^m}^{2^m} |g^k(x)|^2 dx \leq \int_{-2^m}^{2^m} |f(x)|^2 dx < \infty$$

By (7.3) in Lemma 7.2, we shall write

$$\begin{aligned} [F(A^k), F(A^k)]_{t \wedge S_m} &= \int_{-2^m}^{2^m} |g^k(x)|^2 \left[ 2^{-k} \eta^{k,m}(t, x) - 2L_{t \wedge S_m}^x \right] dx \\ &\quad + \int_{-2^m}^{2^m} |g^k(x)|^2 2L_{t \wedge S_m}^x dx = J_1^{k,m}(t) + J_2^{k,m}(t). \end{aligned}$$

By applying Lemma 7.3 and (7.7) it follows that

$$\mathbb{E}|J_1^{k,m}(t)| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

In order to evaluate the limit in the weak sense of the second term, we take an arbitrary  $g \in L^\infty$  and let us denote  $\phi_{t,m}(x) := \mathbb{E}[gL_{t \wedge S_m}^x]$ . We claim that  $\{|g^k|^2 \phi_{t,m}; k \geq 1\}$  is uniformly integrable in  $L^1[-2^m, 2^m]$ . In fact, if  $\Lambda$  is any Borel set in  $[-2^m, 2^m]$  then the continuity of the local time in space gives

$$\begin{aligned} \int_{\Lambda} |g^k(x)|^2 |\phi_{t,m}(x)| dx &\leq C \int_{\Lambda} |g^k(x)|^2 dx \\ &\leq C \int_{\Lambda} |f(x)|^2 dx \quad \forall k \geq 1. \end{aligned}$$

This proves our claim. In this way, we do have  $(|g^k|^2 - |f|^2)\phi_{t,m} \rightarrow 0$  in  $L^1[-2^m, 2^m]$  as  $k \rightarrow \infty$  and therefore

$$\left| \int_{-2^m}^{2^m} [|g^k(x)|^2 - |f(x)|^2] \phi_{t,m}(x) dx \right| \leq \int_{-2^m}^{2^m} ||g^k(x)|^2 - |f(x)|^2| |\phi_{t,m}(x)| dx \rightarrow 0.$$

This proves that  $\mathbb{E}J_2^{k,m}(t) \rightarrow \mathbb{E} \int_{-2^m}^{2^m} |f(x)|^2 g 2L_{t \wedge S_m}^x dx$  as  $k \rightarrow \infty$  for every  $g \in L^\infty$  and therefore it follows that

$$\langle F(B), F(B) \rangle_{t \wedge S_m}^\delta = \int_{-2^m}^{2^m} |f(x)|^2 g 2L_{t \wedge S_m}^x dx = \int_0^{t \wedge S_m} |f(B_s)|^2 ds.$$

This concludes the proof of the Lemma.  $\square$

We are now able to show an intrinsic approximation for the local-time integrals when  $f \in L^2(\mathbb{R})$ . The proof is a simple combination of our results with the ones given in [20, 18].

**Proposition 7.1.** *Assume that  $F$  is absolutely continuous with derivative  $f \in L_{loc}^2(\mathbb{R})$  then*

$$F(B) = F(y) + \int_0^t f(B_s) dB_s - \frac{1}{2} \int_{\mathbb{R}} f(x) d_x L_t^x.$$

Moreover, if  $f \in L^2(\mathbb{R})$  then the following approximation holds

$$(7.8) \quad - \int_{-\infty}^{\infty} f(x) d_x L_t^x = \lim_{k \rightarrow \infty} \frac{1}{2^{-2k}} \sum_{i=-1}^1 \int_0^t \{F(A_{s-}^k + i2^{-k}) - F(A_{s-}^k)\} h_s^k ds \quad \text{weakly in } B^1.$$

*Proof.* The proof is a simple consequence of the previous results together with Prop. 1.4 in [20] and Th.3.1 in [18]. One can easily check that  $F(B)$  satisfies condition (BUI) (locally) if  $f \in L_{loc}^2(\mathbb{R})$ . Therefore, Lemmas 7.4, 7.5 and Theorems 3.1 allow us to state that  $F(B)$  is a local  $\delta$ -weak Dirichlet process and therefore there exists a unique decomposition

$$F(B) = F(y) + M + N,$$

where  $M \in H_{loc}^2$  and  $N$  is  $\delta$ -locally orthogonal to  $H^2$ . Corollary 3.2, the expression (7.5), the orthogonality of  $N$  and Theorem 5.1 yield  $M_t = \int_0^t f(B_s)dB_s$ . By comparing the above decomposition with Prop.1.4 in [20] and Th. 3.1 in [18], we shall conclude from (7.1) in Lemma 7.1 and Theorem 3.1 that the convergence given in (7.8) holds.  $\square$

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